7. LINEAR SYSTEMS OF EQUATIONS

$f_i(x_j) = 0 \quad i = 1,\dots,m \quad j = 1,\dots,n$

IF f_i ARE NON-LINEAR: NO GENERAL METHOD TO SOLVE, METHOD VARIES FROM CASE TO CASE.

HERE: $f_i(x_j) = \sum_j a_{ij} x_j - b_i \qquad a_{ij}, b_i \in \mathbb{R}$ OR $\mathbb{C}$

LINEAR SYSTEM OF EQUATIONS

$m = n$ AND $\det(A) \neq 0 \quad (A = (a_{ij})) \Rightarrow$ UNIQUE SOLUTION.

Jan 21-10:15

7.1 NAIVE GAUSS-ELIMINATION

$A x = b$ $\quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

MATRIX NOTATION

IDEA OF GAUSS-ELIMINATION: FORM LINEAR COMBINATIONS OF EQUATIONS.

$\begin{matrix} f_1(x) = b_1 \\ f_2(x) = b_2 \end{matrix} \Rightarrow f_1(x) + c \cdot f_2(x) = b_1 + c \cdot b_2$ FOR $c \in \mathbb{R}$ OR $\mathbb{C}$

EXAMPLE:

AFTER GAUSS-ELIMINATION, THERE ARE ZEROS BELOW THE DIAGONAL:

Jan 21-10:24

$-3x_4 = -3 \quad \Rightarrow x_4 = 1$

$2x_3 = -9 + 5x_4 \quad \Rightarrow x_3 = -2$

$-4x_2 = -6 - 2x_3 - 2x_4 \quad \Rightarrow x_2 = 1$

$6x_1 = 16 + 2x_2 - 2x_3 - 4x_4 \quad \Rightarrow x_1 = 3$

BACK SUBSTITUTION

MATLAB: `>> x = A\b` ("slash")

FORMULA FOR BACKSUBSTITUTION:

$$x_k = \frac{b_k - \sum_{j=k+1}^{n} B_{kj} x_j}{B_{kk}} \qquad k = n, n-1, \dots, 1$$

Jan 21-10:30

REMARKS:

1) PROGRAM gaussel.m WORKS FOR SEVERAL RIGHT-HAND SIDES ($N > n+1$): FOR EXAMPLE $b = I_{n\times n}$ ($n\times n$ IDENTITY MATRIX)

$A x = b = I_{n\times n} \Rightarrow x = A^{-1}$ (MATRIX INVERSE OF A) !

2) NUMERICAL ACCURACY?

$A = \mathrm{rand}(n,n) \qquad x = \mathrm{rand}(n,1) \quad \rightarrow$ COMPUTE $b = A \cdot x$

SOLVE $A \cdot y = b$ NUMERICALLY FOR y $\begin{cases} \text{gaussel}(A,x) \\ A \backslash b \text{ MATLAB} \end{cases}$

$\delta = \max_i |x_i - y_i|$

→ MATLAB ERRORS ARE SMALLER (FACTOR 10–100)

→ GAUSS-ELIMINATION WITH PIVOTING

Jan 21-11:05

7.2 PIVOTING

SO FAR:

* n EQUATIONS CAN BE WRITTEN IN ARBITRARY ORDER
* $a_{11} = 0$ IS POSSIBLE IN NON-SINGULAR PROBLEMS $Ax = b$ (BUT OUR PROGRAM WOULD CRASH "$1/0$")

⇒ PIVOTING:

- PARTIAL PIVOTING: EXCHANGE THE EQUATIONS (THE ROWS)
- FULL PIVOTING: ALSO EXCHANGE THE COLUMNS (⇒ ORDERING OF THE x_i'S CHANGES!)

PARTIAL PIVOTING

START WITH THE ROW (BECOMES FIRST ROW) WITH $|a_{i1}|$ MAXIMAL

→ PIVOT ELEMENT OR PIVOT (IF $|a_{i1}| = 0 \ \forall i \Rightarrow A$ SINGULAR: $\det A = 0$)

Jan 21-11:20

THEN TAKE THE ROW WHERE $|a_{i2}|$ MAXIMAL ETC.

EXCHANGE OF ROWS:

IN PRACTICE: DO NOT SWAP ROWS IN MEMORY, BUT MAKE INSTEAD A BOOKKEEPING OF THE ROW EXCHANGES = PERMUTATIONS.

VECTOR $p = [1 \ 2 \ \dots \ n]$

EXCHANGE THE COMPONENTS OF p! → GENERATES A PERMUTATION

$[p(1) \ p(2) \ \dots \ p(n)]$

TO ADDRESS ROW i USE $B(p(i), :)$

Jan 21-11:31

7.3 ITERATIVE IMPROVEMENT OF SOLUTION

$A x = b \rightsquigarrow$ SOLUTION $x^{(0)}$

ROUNDOFF ERROR : NUMERICALLY $A x^{(0)} = \tilde{b} \neq b$

LET $A x^* = b$ BE THE EXACT SOLUTION , $\delta x = x^{(0)} - x^*$:

$$\underline{A \cdot \delta x} = A\left(x^{(0)} - x^*\right) = A x^{(0)} - A x^* = \tilde{b} - b =: \underline{\delta b}$$

$\rightsquigarrow$ SOLVE $A \cdot \delta x = \delta b$ FOR δx

$\rightsquigarrow$ CORRECT (IMPROVE) SOLUTION $x^{(1)} = x^{(0)} - \delta x$

THIS CAN BE ITERATED : COMPUTE $\tilde{b} = A \cdot x^{(1)}$ ETC.

Jan 21-11:40