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1 Some differential geometry basics

1.1 Vectors, covectors and tensors

We start with a curved manifold γ. Let $x^\alpha(\lambda)$ be a curve on γ which is parameterized by λ. A scalar field $f(x)$ changes along the curve according to

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{\partial f}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$$

We define

$$f_{,\mu} := \frac{\partial f}{\partial x^\mu} \qquad u^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$$

so

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = f_{,\mu}\mathrm{d}x^\mu$$

The vector u^μ is tangent to the curve (see fig. 1).

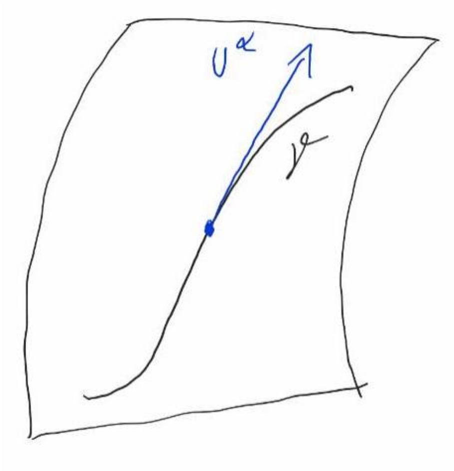


Figure 1. A tangent vector u^μ to the curve γ.

We call $f_{,\mu}$ a covector or dual vector. Vectors and covectors are defined via their transformation properties. In an arbitrary different coordinate system with coordinates $x^{\mu'}$ we can see that

$$f_{,\mu'} = \frac{\partial f}{\partial x^{\mu'}} = \frac{\partial f}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu'}} = f_{,\mu}\frac{\partial x^\mu}{\partial x^{\mu'}}$$

whereas

$$u^{\mu'} = \frac{\mathrm{d}x^{\mu'}}{\mathrm{d}\lambda} = \frac{\partial x^\mu}{\partial \lambda}\frac{\partial x^{\mu'}}{\partial x^\mu} = u^\mu\frac{\partial x^{\mu'}}{\partial x^\mu}$$

Every quantity A^μ that transforms as

$$A^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}A^\mu$$

we call a vector and every quantity B_μ that transforms as

$$B_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}B_\mu$$

we call a covector or dual vector. Note, that the product of a vector with a covector

$$B_{\mu'}A^{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}B_\mu\frac{\partial x^{\mu'}}{\partial x^\mu}A^\mu = B_\mu A^\mu$$

is a scalar. There can be objects that carry more than one index. If each of their indices transforms as either a vector or covector, these objects are called tensors. E.g. an object $C_{\alpha\beta}{}^\gamma$ that transforms as

$$C_{\alpha'\beta'}{}^{\gamma'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^{\gamma'}}{\partial x^\gamma}C_{\alpha\beta}{}^\gamma$$

is called a rank 3 tensor (3 indices) with two covariant indices and one contravariant index. There is one rank two tensor of immediate interest: the metric tensor $g_{\mu\nu}$. It transforms vectors to covectors (it "lowers indices") in the sense that

$$A_\mu := g_{\mu\nu}A^\nu$$

Its inverse is denoted by $g^{\mu\nu}$ and is also called the metric tensor. Since it is the inverse of $g_{\mu\nu}$, we have

$$g_{\mu\nu}g^{\nu\sigma} = \delta_\mu{}^\sigma$$

One important thing to remember is that vectors themselves do not live in the manifold (they live in its tangent space). Just think of a tangent vector to the surface of the sphere to get the idea.

1.2 The covariant derivative

Let us now define the basic concepts that will allow us to do physics. We obviously need to compare various quantities along the curve γ - like the scalar field f we had previously. But we will need to also compare vectors, so let us define a vector field A^μ and let us compare the vector $A^\mu(P)$ with the vector $A^\mu(Q)$ on two points P anf Q on the curve γ (see fig. 2).

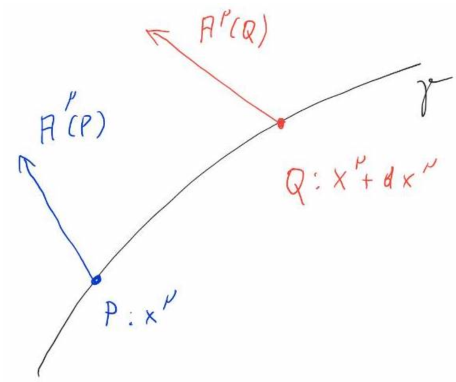


Figure 2. Comparing two vectors along a curve.

If we assume that $P\colon x^\mu$ and $Q\colon x^\mu + \mathrm{d}x^\mu$ are infinitesimally close, we can compute the naive difference

$$\begin{aligned} \mathrm{d}A^\alpha &= A^\alpha(Q) - A^\alpha(P) \\ &= A^\alpha(x^\mu + \mathrm{d}x^\mu) - A^\alpha(x^\mu) \\ &= A^\alpha{}_{,\mu}\mathrm{d}x^\mu \end{aligned} \tag{1}$$

In a dfferent coordinate system, we otain

$$\begin{aligned} A^{\alpha'}{}_{,\mu'} &= \frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\alpha'}}{\partial x^\alpha}A^\alpha\right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\alpha'}}{\partial x^\alpha}A^\alpha\right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}A^\alpha{}_{,\mu} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\mu \partial x^\alpha}A^\alpha \end{aligned} \tag{2}$$

We can see that the derivative $A^\alpha{}_{,\mu}$ does not transform as a tensor. The first term in the last line of (2) alone would be the proper transformation for a tensor. The second term however spoils this, and that should actually not come as a surprise. We are comparing tangent vectors at different points, so they don't even live in the same (tangent-)space. In order to compare them properly, we have to say how the two tangent spaces should be mapped onto each other to make a comparison possible (the usual terminology is: how to transport one vector to the place of the other). If we properly transport the vector A^μ from P to Q then the derivative should transform as a covariant index - just like it does for scalars (which don't have this problem as they have no orientation).

Let us now assume that we have a procedure to transport $A^\alpha(Q)$ back to the point P, resulting in $A^\alpha_T(P)$. We can then uniquely compare it to $A^\alpha(P)$ and obtain

$$\mathrm{D}A^\alpha = A^\alpha_T(P) - A^\alpha(P) = A^\alpha{}_{;\mu}\mathrm{d}x^\mu$$

where we have implicitly defined the covariant derivative $A^\alpha{}_{;\mu}$. Comparing to (1), we write the difference between the procedures

$$\delta A^\alpha = \mathrm{D}A^\alpha - \mathrm{d}A^\alpha = (A^\alpha{}_{;\mu} - A^\alpha{}_{,\mu})\mathrm{d}x^\mu$$

The bracketed expression should be proportional to the vector itself, so we can write quite generically

$$A^\alpha{}_{;\mu} - A^\alpha{}_{,\mu} = \Gamma^\alpha{}_{\beta\mu}A^\beta$$

or

$$A^\alpha{}_{;\mu} = A^\alpha{}_{,\mu} + \Gamma^\alpha{}_{\beta\mu}A^\beta$$

where the $\Gamma^\alpha{}_{\mu\beta}$ are some of yet undetermined coefficients that are called the Christoffel symbols. Before we determine them using some physics arguments, let us see some more mathematical properties they have. First of all, $A^\alpha{}_{;\mu}$ transforms as a second rank tensor with one covariant and one contravariant index. As a derivative it needs to conform to the product rule, so

$$\begin{aligned}(A^\alpha B_\alpha)_{;\mu} &= A^\alpha{}_{;\mu}B_\alpha + A^\alpha B_{\alpha;\mu} \\ &= A^\alpha{}_{,\mu}B_\alpha + \Gamma^\alpha{}_{\beta\mu}A^\beta B_\alpha + A^\alpha B_{\alpha;\mu}\end{aligned}$$

But since the left hand side is a scalar, we also have

$$\begin{aligned}(A^\alpha B_\alpha)_{;\mu} &= (A^\alpha B_\alpha)_{,\mu} \\ &= A^\alpha{}_{,\mu}B_\alpha + A^\alpha B_{\alpha,\mu}\end{aligned}$$

Putting this together we see that

$$A^\alpha B_{\alpha;\mu} = A^\alpha B_{\alpha,\mu} - A^\beta\Gamma^\alpha{}_{\beta\mu}B_\alpha$$

Renaming the indices we have

$$A^\alpha B_{\alpha;\mu} = A^\alpha B_{\alpha,\mu} - A^\alpha\Gamma^\beta{}_{\alpha\mu}B_\beta$$

and since this is true for any A^α we conclude that

$$B_{\alpha;\mu} = B_{\alpha,\mu} - \Gamma^\beta{}_{\alpha\mu}B_\beta \tag{3}$$

Similarly one can find that for higher rank tensors we have to add a term of the form (2) for each contravariant and of the form (3) for each covariant index. For a mixed rank 2 tensor we have e.g.

$$T^\alpha{}_{\beta;\mu} = T^\alpha{}_{\beta,\mu} + \Gamma^\alpha{}_{\gamma\mu}T^\gamma{}_\beta - \Gamma^\gamma{}_{\beta\mu}T^\alpha{}_\gamma$$

Now let us turn to the question we have left open: What is Γ and how can we compute it? For this we first note that the equivalence principle implies that the Christoffel symbols are symmetric in their two lower indices $\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$ and that the covariant derivative is metric compatible in the sense that its covariant derivative vanishes $g_{\mu\nu;\alpha} = 0$. From these two properties it is easy to show that, in terms of the metric, the Christoffel symbols are given as

$$\boxed{\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu})} \tag{4}$$

Exercise 1. Show the preceeding statements, i.e. why the equivalence principle implies symmetry and metric compatibility of $\Gamma^\alpha{}_{\beta\gamma}$ and how one can obtain from this (4).

1.3 Geodesics

A geodesic extremises the distance between two points. The relevance for GR comes from the fact that in a fixed gravitational field (i.e. a fixed metric) test particles move along geodesics. This is the direct realization of the equivalence principle.

The invariant distance between two points infinitesimally separated by $\mathrm{d}x^\mu$ is

$$\mathrm{d}s^2 = \mathrm{d}x^\alpha \mathrm{d}x_\alpha = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = g_{\alpha\beta}u^\alpha u^\beta \mathrm{d}\lambda^2 \tag{5}$$

The invariant distance s between two points P and Q on the manifold (i.e. two events in spacetime) is thus given by

$$s = \int_P^Q \sqrt{\pm g_{\alpha\beta}u^\alpha u^\beta}\,\mathrm{d}\lambda$$

where the sign inside the square root is set such that the argument is positive. We extremise s by demanding that its variation vanishes (as in classical mechanics). Remembering that $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ and defining

$$L = \sqrt{\pm g_{\alpha\beta}u^\alpha u^\beta}$$

we find from a straightforward application of variational calculus that

$$\begin{aligned}
\delta s &= \int_P^Q \left(\frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial u^\mu}\delta u^\mu\right)\mathrm{d}\lambda \\
&= \int_P^Q \left(\frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial u^\mu}\delta x^\mu\right) - \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial u^\mu}\right)\delta x^\mu\right)\mathrm{d}\lambda \\
&= \frac{\partial L}{\partial u^\mu}\delta x^\mu\bigg|_P^Q + \int_P^Q \left(\frac{\partial L}{\partial x^\mu}\delta x^\mu - \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial u^\mu}\right)\delta x^\mu\right)\mathrm{d}\lambda
\end{aligned}$$

Since the initial points P and Q are fixed, the variation δx^μ vanishes there and so does the first term in the last line. If we thus want impose $\delta s = 0$, the second term has to vanish and, since it has to do so for arbitrary δx^μ, the integrand has to vanish. This of course implies the well known Euler-Lagrange equation

$$\frac{\partial L}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial u^\mu}$$

In our case we have

$$\frac{\partial L}{\partial x^\mu} = \frac{\pm g_{\alpha\beta,\mu}}{2L}u^\alpha u^\beta$$

and

$$\frac{\partial L}{\partial u^\mu} = \frac{\pm(g_{\alpha\mu} + g_{\mu\alpha})}{2L}u^\alpha = \frac{\pm g_{\alpha\mu}}{L}u^\alpha$$

so

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial u^\mu} = \pm\frac{1}{L}\left(u^\alpha \underbrace{\frac{\mathrm{d}}{\mathrm{d}\lambda}g_{\alpha\mu}}_{g_{\alpha\mu,\beta}u^\beta} + g_{\alpha\mu}\frac{\mathrm{d}}{\mathrm{d}\lambda}u^\alpha - \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}\lambda}g_{\alpha\mu}u^\alpha\right)$$

and the Euler-Lagrange equation reads

$$\frac{1}{2}g_{\alpha\beta,\mu}u^\alpha u^\beta = g_{\alpha\mu,\beta}u^\alpha u^\beta + g_{\alpha\mu}\frac{\mathrm{d}}{\mathrm{d}\lambda}u^\alpha - \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}\lambda}g_{\alpha\mu}u^\alpha$$

Multiplying with $g^{\mu\nu}$, rearranging the terms and defining

$$\dot{a} := \frac{\mathrm{d}a}{\mathrm{d}\lambda} \qquad \kappa = \frac{\dot{L}}{L}$$

we obtain

$$\dot{u}^\nu + \underbrace{\frac{1}{2}g^{\mu\nu}(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu})}_{\Gamma^\nu{}_{\alpha\beta}}u^\alpha u^\beta = \kappa u^\nu$$

which is called the geodesic equation. In its final form it reads

$$\boxed{\dot{u}^\nu + \Gamma^\nu{}_{\alpha\beta}u^\alpha u^\beta = \kappa u^\nu}$$

After extremisation, we may choose any parameter λ along our curve to parameterize it. If the curve is timelike ($s<0$), we can choose the proper time and if it is spacelike ($s>0$), we may choose the proper distance s itself. In the latter case we obviously have $\mathrm{d}s^2 = \mathrm{d}\lambda^2$ and thus, according to (5) $g_{\alpha\beta}u^\alpha u^\beta = 1$, which in turn implies $L=1$ and $\dot{L} = \kappa = 0$. In the former case, we have $\mathrm{d}s^2 = -\mathrm{d}\lambda^2$, which implies $g_{\alpha\beta}u^\alpha u^\beta = -1$ and also $L=1$ and $\dot{L} = \kappa = 0$. The only difference between the cases is the sign needed to render the argument of the square root positive. When $\kappa = 0$ we call the parameterization affine and the geodesic equation reduces to

$$\boxed{\dot{u}^\nu + \Gamma^\nu{}_{\alpha\beta}u^\alpha u^\beta = 0}$$

We can recast this equation by noticing that

$$\dot{u}^\nu = u^\nu{}_{,\alpha}u^\alpha$$

so that in total we have

$$(u^\nu{}_{,\alpha} + \Gamma^\nu{}_{\alpha\beta}u^\beta)u^\alpha = u^\nu{}_{;\alpha}u^\alpha = 0$$

We can define the covariant derivative along the geodesic

$$D_\lambda u^\nu = u^\nu{}_{;\alpha}u^\alpha$$

so that the affinely parameterised geodesic equation ultimately reads

$$D_\lambda u^\nu = 0$$

If the geodesic is lightlike ($s=0$), then an affine parameterisation can not be directly found in terms of the eigentime or proper distance, because both vanish. One can however find an affine parameterisation λ^* from a non-affine one λ, characterized by $\kappa(\lambda)$, if one takes a λ^* fulfilling

$$\frac{\mathrm{d}\lambda^*}{\mathrm{d}\lambda} = e^{\int^\lambda \mathrm{d}\lambda' \kappa(\lambda')}$$

Exercise 2. Prove this statement.

It is also interesting that the tangent vector u^μ to an affinely parameterised geodesic has always unit norm. This can be seen rather easily:

$$\begin{aligned}\frac{\mathrm{d}(u^\mu u_\mu)}{\mathrm{d}\lambda} &= (u^\mu u_\mu)_{;\nu} u^\nu \\ &= u_\mu \underbrace{u^\mu{}_{;\nu} u^\nu}_{0} + u^\mu \underbrace{u_{\mu;\nu} u^\nu}_{0} \\ &= 0\end{aligned}$$

1.4 The Lie derivative

One important question on a curved manifold is how to recognize a symmetry. Since we do not have the usual global tools available (what e.g. is a shift in time?), we need to recognize them locally. For this purpose we first need to introduce the concept of Lie derivative, which is distinct from the covariant derivative and in some sense simpler.

Let us start by tracing out a path $x^\mu(\lambda)$ in our manifold which is parameterized by λ. Then we pick an arbitrary point $x^\mu(\lambda)$ along this path and one point that is infinitesimally close to it

$$y^\mu = x^\mu(\lambda + \mathrm{d}\lambda) = x^\mu(\lambda) + \xi^\mu \mathrm{d}\lambda \qquad \xi^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$$

We now look at an arbitrary vector field $A^\alpha(x)$ at those two points. One way of looking at it is that $x = x(\lambda)$ and y are just separated by an infinitesimal change in coordinate and thus

$$A^\alpha(y) = A^\alpha(x + \xi \mathrm{d}\lambda) = A^\alpha(x) + \mathrm{d}\lambda \xi^\mu A^\alpha{}_{,\mu}(x)$$

On the other hand, we can treat $A^\alpha(y)$ and $A^\alpha(x)$ as related by a coordinate *transformation*

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu + \xi^\mu{}_{,\nu} \mathrm{d}\lambda$$

and thus the vector field A^μ in the x-basis transforms to $\hat{A}^\alpha$ in the y basis according to the usual transformation law

$$\hat{A}^\alpha = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) A^\mu = A^\alpha + A^\mu \xi^\alpha{}_{,\mu} \mathrm{d}\lambda$$

The difference between these two interpretations defines the Lie derivative

$$\mathcal{L}_\xi A^\alpha = \frac{A^\alpha(y) - \hat{A}^\alpha}{\mathrm{d}\lambda} = \xi^\mu A^\alpha{}_{,\mu} - A^\mu \xi^\alpha{}_{,\mu} \tag{6}$$

Note that we can write this in terms of covariant derivatives as well

$$\begin{aligned}\mathcal{L}_\xi A^\alpha &= \xi^\mu A^\alpha{}_{,\mu} - A^\mu \xi^\alpha{}_{,\mu} \\ &= \xi^\mu A^\alpha{}_{;\mu} - \xi^\mu \Gamma^\alpha_{\beta\mu} A^\beta - A^\mu \xi^\alpha{}_{;\mu} + A^\mu \Gamma^\alpha_{\beta\mu} \xi^\beta \\ &= \xi^\mu A^\alpha{}_{;\mu} - A^\mu \xi^\alpha{}_{;\mu}\end{aligned}$$

Also there is the obvious antisymmetry

$$\mathcal{L}_\xi A^\alpha = -\mathcal{L}_A \xi^\alpha$$

For a covector B_α we can carry through the same procedure and obtain

$$\hat{B}_\alpha = \left(\frac{\partial x^\mu}{\partial y^\alpha}\right) B_\mu = B_\alpha - \xi^\mu{}_{,\alpha} B_\mu \mathrm{d}\lambda$$

and thus

$$\mathcal{L}_\xi B_\alpha = \xi^\mu B_{\alpha,\mu} + \xi^\mu{}_{,\alpha} B_\mu$$

It is easy to see that we can rewrite this with covariant derivatives, too

$$\begin{aligned}
\mathcal{L}_\xi B_\alpha &= \xi^\mu B_{\alpha,\mu} + \xi^\mu{}_{,\alpha} B_\mu \\
&= \xi^\mu B_{\alpha;\mu} - \xi^\mu \Gamma^\beta_{\alpha\mu} B_\beta + \xi^\mu{}_{;\alpha} B_\mu + \Gamma^\mu_{\beta\alpha} \xi^\beta B_\mu \\
&= \xi^\mu B_{\alpha;\mu} + \xi^\mu{}_{;\alpha} B_\mu
\end{aligned}$$

The Lie derivative of higher rank tensors always has the first term, the derivative of the tensor, and one additional term with the derivative of ξ contracted with each tensor index. For a rank 2 covariant tensor we have e.g.

$$\mathcal{L}_\xi g_{\alpha\beta} = \xi^\mu g_{\alpha\beta;\mu} + \xi^\mu{}_{;\alpha} g_{\mu\beta} + \xi^\mu{}_{;\beta} g_{\alpha\mu} \tag{7}$$

or, with ordinary derivatives,

$$\mathcal{L}_\xi g_{\alpha\beta} = \xi^\mu g_{\alpha\beta,\mu} + \xi^\mu{}_{,\alpha} g_{\mu\beta} + \xi^\mu{}_{,\beta} g_{\alpha\mu}$$

Finally a bit of nomenclature: If the Lie derivative of an object with respect to the vector ξ vanishes, it is said to be Lie transported by ξ.

1.5 Killing vectors

If an object is Lie transported by a vector field ξ, it is unable to distinguish whether ξ was due to a simple change in coordinates or a coordinate transformation. In other words, that object can't distinguish the original manifold from the one produced by the infinitesimal shift with the vector field ξ. If the metric can not distinguish such a shift, none of the physics will be able to and we have a symmetry. This is the basic motivation behind the definition of a Killing vector field ξ

$$\mathcal{L}_\xi g_{\alpha\beta} = 0$$

Using (7) and the metric compatibility of the covariant derivative this can be rewritten as

$$\xi^\mu{}_{;\alpha}\, g_{\mu\beta} + \xi^\mu{}_{;\beta}\, g_{\alpha\mu} = 0$$

or

$$\xi_{\beta;\alpha} + \xi_{\alpha;\beta} = 0$$

which is known as the Killing equation. The importance of the Killing vector and its precise meaning as a statement of symmetry is exposed by looking at an affinely parameterised geodesic $x^\mu(\lambda)$ with a tangent (velocity) $u^\mu = \mathrm{d}x^\mu(\lambda)/\mathrm{d}\lambda$. When we take the product of a Killing vector ξ^μ with the velocity, we find that its (covariant) derivative along the geodesic vanishes

$$\begin{aligned}
D_\lambda(\xi_\alpha u^\alpha) &= (\xi_\alpha u^\alpha)_{;\beta} u^\beta \\
&= \xi_{\alpha;\beta} u^\beta u^\alpha + \xi_\alpha \underbrace{u^\alpha{}_{;\beta} u^\beta}_{0} \\
&= \frac{1}{2}(\xi_{\alpha;\beta} + \xi_{\beta;\alpha}) u^\alpha u^\beta \\
&= \frac{1}{2}(\xi_{\alpha;\beta} - \xi_{\alpha;\beta}) u^\alpha u^\beta \\
&= 0
\end{aligned}$$

The second term in the second line vanishes by the geodesic equation while in the third line we have used the Killing equation to obtain an expression that is both odd and even under the exchange $\alpha \leftrightarrow \beta$, so that it also vanishes. The quantity $\xi_\alpha u^\alpha$ is thus conserved - another deep relation between symmetry and conservation laws.

1.6 The invariant volume element and the metric determinant

The coordinate transformation

$$x^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{a}} x^{\alpha}$$

has a Jacobian

$$J = \left| \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \right|$$

so that the invariant volume element transforms as

$$\mathrm{d}^4 x = J \mathrm{d}^4 x'$$

The metric itself transforms as

$$g_{\alpha'\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} g_{\alpha\beta}$$

so that its determinant transforms as

$$|g_{\alpha'\beta'}| = \left| \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} g_{\alpha\beta} \right| = J^2 |g_{\alpha\beta}|$$

In a local Lorentz frame the metric is the simple minkovski metric $\eta_{\alpha\beta}$ which has determinant $|\eta_{\alpha\beta}| = -1$. We now want to construct an invariant infinitesimal 4-volume element that is locally equivalent to the 4-volume in the local Lorentz frame. We can achieve this by taking $g_{\alpha\beta} = \eta_{\alpha\beta}$. Then we have

$$|g_{\alpha'\beta'}| = J^2 |g_{\alpha\beta}| = -J^2$$

or

$$J = \sqrt{-|g_{\alpha'\beta'}|}$$

and thus we obtain an invariant volume element that is locally equivalent to the volume element of the local Lorentz frame

$$\mathrm{d}^4 x = J \mathrm{d}^4 x' = \sqrt{-|g_{\alpha'\beta'}|} \mathrm{d}^4 x'$$

It is customary to abbreviate the determinant of the metric tensor as

$$g = |g_{\alpha'\beta'}|$$

so the invariant 4-volume element reads

$$\sqrt{-g}\, \mathrm{d}^4 x'$$

The metric determinant occurs in a very useful relation, the divergence formula. The divergence formula states that for any vector A^α the covariant derivative contracted with the vector is

$$A^\nu{}_{;\nu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^\nu)_{,\nu}$$

To prove this relation we first note that the variation of the log of the determinant of an arbitrary matrix M with respect to a variation of the matrix δM might generically be written as

$$\begin{aligned} \delta \ln||M|| &= \ln||M+\delta M|| - \ln||M|| \\ &= \ln|M^{-1}(M+\delta M)| \\ &= \ln|\mathbb{1} + M^{-1}\delta M| \\ &= \mathrm{Tr}(\ln(\mathbb{1} + M^{-1}\delta M)) \\ &= \mathrm{Tr}(M^{-1}\delta M) + O((\delta M)^2) \end{aligned}$$

We apply this to the metric tensor and obtain to leading order

$$\delta \ln|g| = g^{\alpha\beta}\delta g_{\beta\alpha}$$

Remembering that $g<0$ we find

$$(\ln\sqrt{-g})_{,\mu} = \frac{1}{2}\ln(-g)_{,\mu} = \frac{1}{2}g^{\alpha\beta}g_{\beta\alpha,\mu}$$

Because of the metric compatibility of the covariant derivative $g_{\beta\alpha;\mu}=0$, we can simplify this to

$$(\ln\sqrt{-g})_{,\mu} = \frac{1}{2}g^{\alpha\beta}(\Gamma^\nu_{\beta\mu}g_{\nu\alpha} + \Gamma^\nu_{\alpha\mu}g_{\beta\nu}) = \Gamma^\nu_{\nu\mu}$$

Returning to the divergence formula we thus find

$$\begin{aligned} A^\nu{}_{;\nu} &= A^\nu{}_{,\nu} + \Gamma^\nu_{\mu\nu}A^\mu \\ &= A^\nu{}_{,\nu} + A^\mu(\ln\sqrt{-g})_{,\mu} \\ &= \frac{\sqrt{-g}A^\nu{}_{,\nu} + A^\nu(\sqrt{-g})_{,\nu}}{\sqrt{-g}} \\ &= \frac{(\sqrt{-g}A^\nu)_{,\nu}}{\sqrt{-g}} \end{aligned}$$

which proves the relation.

1.7 The totally antisymmetric tensor

Let us start by defining the totally antisymmetric symbol

$$[\alpha\beta\gamma\delta] = \begin{cases} 1 & \text{even permuataion of } 0123 \\ -1 & \text{odd permutation of } 0123 \\ 0 & \text{else} \end{cases}$$

which is not a tensor. We can use this symbol to define the determinant of a matrix,

$$|M| = [\alpha\beta\gamma\delta]M_{0\alpha}M_{1\beta}M_{2\gamma}M_{3\delta}$$

The metric determinant can thus be written as

$$g = [\alpha\beta\gamma\delta]g_{0\alpha}g_{1\beta}g_{2\gamma}g_{3\delta}$$

and the invariant volume element is (note that $g<0$)

$$\sqrt{-g}\,\mathrm{d}^4x = \underbrace{-\frac{1}{\sqrt{-g}}[\alpha\beta\gamma\delta]}_{\varepsilon^{\alpha\beta\gamma\delta}} g_{0\alpha}g_{1\beta}g_{2\gamma}g_{3\delta}\mathrm{d}^4x$$

Since this is an (invariant) scalar, we conclude that

$$\varepsilon^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}}[\alpha\beta\gamma\delta]$$

is a rank 4 contravariant tensor, which we call the Levi-Civita tensor. Its contravariant counterpart is

$$\varepsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}[\alpha\beta\gamma\delta]$$

Note the relative minus sign which we can easily see is necessary, since

$$\begin{aligned}
\varepsilon_{\alpha\beta\gamma\delta} &= g_{\alpha\alpha'}g_{\beta\beta'}g_{\gamma\gamma'}g_{\delta\delta'}\varepsilon^{\alpha'\beta'\gamma'\delta'} \\
&= -\frac{1}{\sqrt{-g}}\underbrace{g_{\alpha\alpha'}g_{\beta\beta'}g_{\gamma\gamma'}g_{\delta\delta'}[\alpha'\,\beta'\,\gamma'\,\delta']}_{g[\alpha\beta\gamma\delta]} \\
&= \sqrt{-g}[\alpha\beta\gamma\delta]
\end{aligned}$$

1.8 Curvature

We now come to the crucial concept of curvature. What we mean by curvature is going around an infinitesimal loop and parallel transporting a vector along that path. If there is curvature, the two vectors will not match, but disagree by a small amount proportional to the vector itself. The constant of proportionality is the curvature.

Let us now formulate this precisely. The small infinitesimal loop is realized by going into two (different) coordinate directions α and β in alternating order, i.e.

$$A^\mu{}_{;\alpha\beta} = A^\mu{}_{;\beta\alpha} - R^\mu{}_{\nu\alpha\beta}A^\nu$$

In addition we have the original direction μ of the vector and the direction ν in which the change of the vector points. The proportionality constant, i.e. the curvature, therefore needs to be a 4 index object $R^\mu{}_{\nu\alpha\beta}$, the Riemann curvature tensor. From this definition we can also obtain an explicit expression for the curvature tensor in terms of the Christoffel symbols as

$$\begin{aligned}
R^\mu{}_{\nu\alpha\beta}A^\nu &= A^\mu{}_{;\beta\alpha} - A^\mu{}_{;\alpha\beta} \\
&= A^\mu{}_{;\beta,\alpha} + \Gamma^\mu_{\nu\alpha}A^\nu{}_{;\beta} - \Gamma^\nu_{\beta\alpha}A^\mu{}_{;\nu} - A^\mu{}_{;\alpha,\beta} - \Gamma^\mu_{\nu\beta}A^\nu{}_{;\alpha} + \Gamma^\nu_{\alpha\beta}A^\mu{}_{;\nu} \\
&= A^\mu{}_{;\beta,\alpha} + \Gamma^\mu_{\nu\alpha}A^\nu{}_{;\beta} - A^\mu{}_{;\alpha,\beta} - \Gamma^\mu_{\nu\beta}A^\nu{}_{;\alpha} \\
&= (A^\mu{}_{,\beta} + \Gamma^\mu_{\sigma\beta}A^\sigma)_{,\alpha} + \Gamma^\mu_{\nu\alpha}(A^\nu{}_{,\beta} + \Gamma^\nu_{\sigma\beta}A^\sigma) - (A^\mu{}_{,\alpha} + \Gamma^\mu_{\sigma\alpha}A^\sigma)_{,\beta} - \Gamma^\mu_{\nu\beta}(A^\nu{}_{,\alpha} + \Gamma^\nu_{\sigma\alpha}A^\sigma) \\
&= (\Gamma^\mu_{\sigma\beta}A^\sigma)_{,\alpha} + \Gamma^\mu_{\nu\alpha}(A^\nu{}_{,\beta} + \Gamma^\nu_{\sigma\beta}A^\sigma) - (\Gamma^\mu_{\sigma\alpha}A^\sigma)_{,\beta} - \Gamma^\mu_{\nu\beta}(A^\nu{}_{,\alpha} + \Gamma^\nu_{\sigma\alpha}A^\sigma) \\
&= \Gamma^\mu_{\sigma\beta,\alpha}A^\sigma + \Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\sigma\beta}A^\sigma - \Gamma^\mu_{\sigma\alpha,\beta}A^\sigma - \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\sigma\alpha}A^\sigma \\
&= (\Gamma^\mu_{\nu\beta,\alpha} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\nu\alpha,\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\nu\alpha})A^\nu
\end{aligned}$$

and thus

$$\boxed{R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\nu\alpha}}$$

From this explicit form one can already see one of the symmetries of the curvature tensor, namely that it is antisymmetric under the exchange of the last two indices $\alpha \leftrightarrow \beta$. There are further symmetries which are best seen when going to locally flat coordinates, i.e. to coordinates where

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + O(x^2)$$

There we have

$$\Gamma^\alpha_{\beta\gamma} \stackrel{*}{=} \frac{1}{2}\eta^{\alpha\mu}(g_{\mu\gamma,\beta} + g_{\beta\mu,\gamma} - g_{\beta\gamma,\mu})$$

Since at the origin the Christoffel symbols vanish $\Gamma^\alpha_{\beta\gamma}(0) \stackrel{*}{=} 0$, we have

$$\begin{aligned} R^\mu{}_{\nu\alpha\beta} &\stackrel{*}{=} \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} \\ &= \frac{1}{2}\eta^{\mu\sigma}(g_{\sigma\nu,\beta\alpha} + g_{\beta\sigma,\nu\alpha} - g_{\beta\nu,\sigma\alpha} - g_{\sigma\nu,\alpha\beta} - g_{\alpha\sigma,\nu\beta} + g_{\alpha\nu,\sigma\beta}) \\ &= \frac{1}{2}\eta^{\mu\sigma}(g_{\beta\sigma,\nu\alpha} - g_{\beta\nu,\sigma\alpha} - g_{\alpha\sigma,\nu\beta} + g_{\alpha\nu,\sigma\beta}) \end{aligned}$$

and thus, in locally flat coordinates,

$$R_{\mu\nu\alpha\beta} \stackrel{*}{=} \frac{1}{2}(g_{\beta\mu,\nu\alpha} + g_{\alpha\nu,\mu\beta} - g_{\beta\nu,\mu\alpha} - g_{\alpha\mu,\nu\beta}) \tag{8}$$

We can now see two additional symmetries of the curvature tensor: First the symmetry under simultaneous exchange of $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$ (implying an antisymmetry under exchange of $\mu \leftrightarrow \nu$). And second the following identity

$$\begin{aligned} R_{\mu\{\nu\alpha\beta\}} &= R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} \\ &= \frac{1}{2}(g_{\beta\mu,\nu\alpha} + g_{\alpha\nu,\mu\beta} - g_{\beta\nu,\mu\alpha} - g_{\alpha\mu,\nu\beta}) \\ &\quad + \frac{1}{2}(g_{\alpha\mu,\beta\nu} + g_{\nu\beta,\mu\alpha} - g_{\alpha\beta,\mu\nu} - g_{\nu\mu,\beta\alpha}) \\ &\quad + \frac{1}{2}(g_{\nu\mu,\alpha\beta} + g_{\beta\alpha,\mu\nu} - g_{\nu\alpha,\mu\beta} - g_{\beta\mu,\alpha\nu}) \\ &= 0 \end{aligned}$$

Finally, there is the very important Bianchi identity

$$\begin{aligned} R_{\mu\nu\{\alpha\beta;\gamma\}} &\stackrel{*}{=} R_{\mu\nu\{\alpha\beta,\gamma\}} \\ &= \frac{1}{2}(g_{\mu\{\beta,\alpha\gamma\}\nu} + g_{\nu\{\alpha,\beta\gamma\}\mu} - g_{\nu\{\beta,\alpha\gamma\}\mu} - g_{\mu\{\alpha,\beta\gamma\}\nu}) \\ &= \frac{1}{2}(g_{\mu\{\beta,\gamma\alpha\}\nu} - g_{\mu\{\alpha,\beta\gamma\}\nu} + g_{\nu\{\alpha,\beta\gamma\}\mu} - g_{\nu\{\beta,\gamma\alpha\}\mu}) \\ &= 0 \end{aligned}$$

Since we have formulated it as a covariant statement

$$R_{\mu\nu\{\alpha\beta;\gamma\}} = 0$$

it is true in any coordinates.

1.9 Ricci tensor and scalar, the Einstein tensor and the field equations

Due to the symmetries of the curvature tensor there is basically a single nontrivial contraction one can perform

$$R_{\alpha\beta} := R^\mu{}_{\alpha\mu\beta}$$

which is called the Ricci tensor. It inherits some symmetries from the curvature tensor, namely

$$\begin{aligned} R_{\alpha\beta} &= g^{\mu\nu} R_{\mu\alpha\nu\beta} \\ &= g^{\mu\nu} R_{\nu\beta\mu\alpha} \\ &= R_{\beta\alpha} \end{aligned}$$

We can contract the Ricci tensor once more to obtain the Ricci scalar

$$R = R^{\alpha}{}_{\alpha}$$

Also, the Bianchi identity implies

$$\begin{aligned} R_{,\sigma} &= R_{;\sigma} \\ &= g^{\mu\alpha} g^{\nu\beta} R_{\mu\nu\alpha\beta;\sigma} \\ &= -g^{\mu\alpha} g^{\nu\beta} (R_{\mu\nu\sigma\alpha;\beta} + R_{\mu\nu\beta\sigma;\alpha}) \\ &= g^{\mu\alpha} g^{\nu\beta} (R_{\mu\nu\alpha\sigma;\beta} + R_{\mu\nu\sigma\beta;\alpha}) \\ &= R^{\beta}{}_{\sigma;\beta} + R^{\alpha}{}_{\sigma;\alpha} \\ &= 2R^{\alpha}{}_{\sigma;\alpha} \end{aligned}$$

which we can write as

$$\delta^{\alpha}_{\sigma} R_{;\alpha} = 2R^{\alpha}{}_{\sigma;\alpha}$$

or

$$\left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right)_{;\alpha} = 0$$

The bracketed expression, which has a vanishing covariant divergence, is called the Einstein tensor

$$G^{\alpha\beta} := R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$$

Since its covariant divergence vanishes, we can equate it to a covariantly conserved physical quantity. Without any physical proof we state that this is proportional to the energy-momentum tensor

$$\boxed{G^{\alpha\beta} = 8\pi T^{\alpha\beta}}$$

where we have set the units such that Newtons constant $G = 1$.

Exercise 3. Show that in the appropriate limits this field equation reduces to Newtonian gravity, i.e. that geodesics are characterized by

$$\frac{\mathrm{d}^2 \vec{x}}{\mathrm{d}t^2} = -\vec{\nabla}\Phi$$

where Φ is the gravitational potential that is related to the density ρ as $\vec{\nabla}^2\Phi = 4\pi\rho$. You can assume that the masses that produce the gravitational field are static.

We can take the trace of this equation to obtain

$$\begin{aligned} 8\pi T^{\alpha}{}_{\alpha} &= G^{\alpha}{}_{\alpha} \\ &= R^{\alpha}{}_{\alpha} - \frac{1}{2} \delta^{\alpha}_{\alpha} R \\ &= -R \end{aligned}$$

Defining the trace of the energy momentum tensor $T = T^\alpha{}_\alpha$ we can thus write the field equations as

$$R^{\alpha\beta} = 8\pi\left(T^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}T\right) \tag{9}$$

1.10 Geodesic deviation

Let us now investigate the behaviour of neighboring geodesics and see how the curvature tensor influences it. Let us take neighboring geodesics γ^s that are affinely parameterised by $x^\alpha(s,t)$ for all s. We can define a tangent vector

$$u^\alpha = \frac{\partial x^\alpha}{\partial t}$$

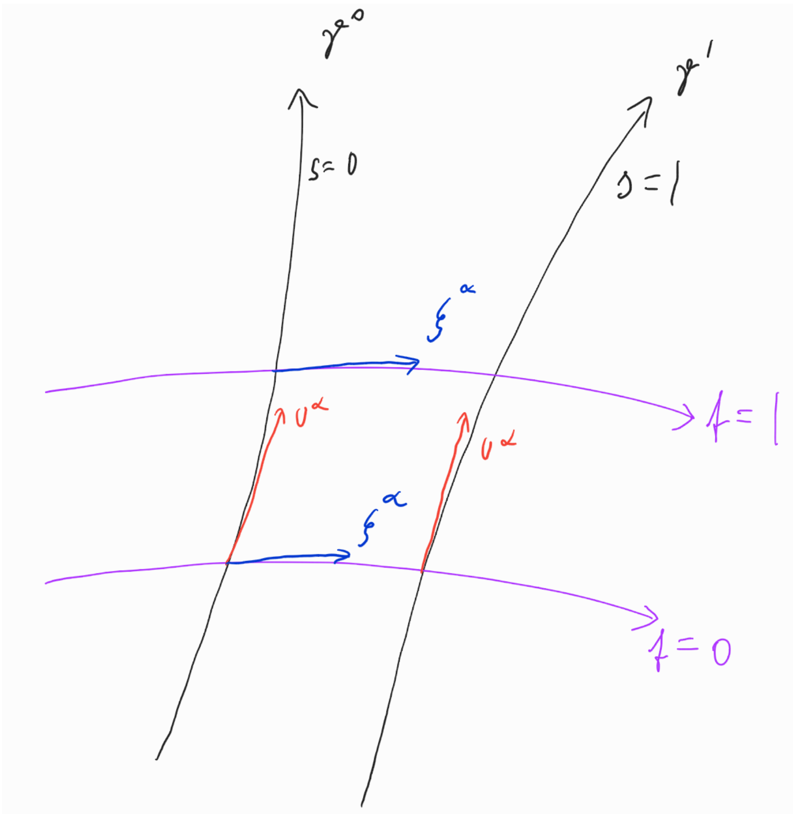
that then fulfills the geodesic equation

$$u^\alpha{}_{;\beta}u^\beta = 0 \tag{10}$$

for any given s. Similarly we can construct tangent vectors

$$\xi^\alpha = \frac{\partial x^\alpha}{\partial s}$$

for equal t. We can picture this as follows:



Since geodesics have constant s, the coordinate difference in s between them is constant. Thus the covariant derivative of the respective tangent vector, i.e. $\frac{\mathrm{D}\xi^\alpha}{\mathrm{d}t}$ is a measure of the velocity with which two such geodesics deviate and the second derivative

$$\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}t^2}$$

describes the acceleration of the geodesics with respect to each other. We can write this quantity along one geodesic as

$$\left.\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}t^2}\right|_{\gamma^s} = (\xi^\alpha{}_{;\beta}u^\beta)_{;\mu}u^\mu|_{\gamma^s}$$

Note that in flat space, where geodesics are straight lines, ξ^α is at most linear in the coordinates and thus the acceleration vanishes. In general however it does not vanish and we want to find an expression for it. We start by noting that since the derivatives with respect to s and t commute, we have

$$\frac{\partial u^\alpha}{\partial s} = \frac{\partial x^\alpha}{\partial s \partial t} = \frac{\partial \xi^\alpha}{\partial t}$$

We can write this equation as

$$\begin{aligned} 0 &= \frac{\partial u^\alpha}{\partial s} - \frac{\partial \xi^\alpha}{\partial t} \\ &= \frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial s} - \frac{\partial \xi^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial t} \\ &= u^\alpha{}_{,\beta}\xi^\beta - \xi^\alpha{}_{,\beta}u^\beta \end{aligned}$$

and identify the right hand side as the Lie derivative (6). Thus $\mathcal{L}_u\xi^\alpha = 0$, which we can write in a covariant fashion as

$$u^\alpha{}_{;\beta}\xi^\beta = \xi^\alpha{}_{;\beta}u^\beta$$

With this relation and (10) we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\xi^\alpha u_\alpha) &= (\xi^\alpha u_\alpha)_{;\beta}u^\beta \\ &= \xi^\alpha{}_{;\beta}u_\alpha u^\beta + \xi^\alpha \underbrace{u_{\alpha;\beta}u^\beta}_{0} \\ &= u^\alpha{}_{;\beta}\xi^\beta u_\alpha \\ &= \frac{1}{2}(u^\alpha u_\alpha)_{;\beta}\xi^\beta \\ &= 0 \end{aligned}$$

Therefore if the tangent vectors are orthogonal initially, they will stay so for all t. We can in fact choose a proper starting point $t=0$ for for the various s such that this condition is always fulfilled and thus we can assume orthogonality in the form $\xi^\alpha u_\alpha = 0$ from now on.

We are now finally in a position to reexpress the geodesic deviation as

$$\begin{aligned} \frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}t^2} &= (\xi^\alpha{}_{;\beta}u^\beta)_{;\mu}u^\mu \\ &= (u^\alpha{}_{;\beta}\xi^\beta)_{;\mu}u^\mu \\ &= u^\alpha{}_{;\beta\mu}\xi^\beta u^\mu + u^\alpha{}_{;\beta}\xi^\beta{}_{;\mu}u^\mu \\ &= u^\alpha{}_{;\mu\beta}\xi^\beta u^\mu - R^\alpha{}_{\nu\beta\mu}u^\nu\xi^\beta u^\mu + u^\alpha{}_{;\beta}u^\beta{}_{;\mu}\xi^\mu \\ &= ((u^\alpha{}_{;\mu}u^\mu)_{;\beta} - u^\alpha{}_{;\mu}u^\mu{}_{;\beta})\xi^\beta - R^\alpha{}_{\nu\beta\mu}u^\nu\xi^\beta u^\mu + u^\alpha{}_{;\mu}u^\mu{}_{;\beta}\xi^\beta \\ &= -R^\alpha{}_{\nu\beta\mu}u^\nu\xi^\beta u^\mu \end{aligned}$$

The resulting equation

$$\boxed{\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}t^2} = -R^\alpha{}_{\beta\mu\nu}u^\beta\xi^\mu u^\nu}$$

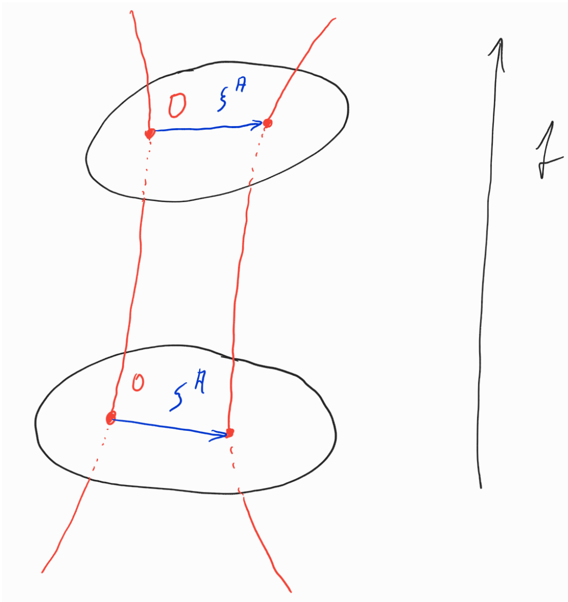
is known as the geodesic deviation equation. It exposes the geometric meaning of curvature as the relative acceleration of geodesics with respect to each other.

1.11 Geodesic congruences

Bundles of geodesics are called congruences. More specifically, a geodesic congruence is a family of geodesics such that only one passes through any point in an open subset of the manifold. The evolution of their cross sections with time gives us a lot of information about the metric they live on.

1.11.1 Linear deformations

We will start out by looking at a two dimensional deformable medium (think of a membrane) and we want to identify observables that tell us whether the medium stretches, contracts and twists. For this purpose, let us define a reference point O on the surface of the medium and let ξ^A be an infinitesimal displacement vector in its neighborhood.

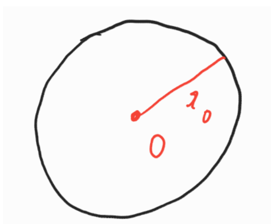


Generically, we can write the time evolution of this displacement vector to leading order as

$$\frac{\mathrm{d}\xi^A}{\mathrm{d}t} = B^A{}_C \xi^C + O(\xi^2) \tag{11}$$

Obviously B encodes the dynamics of the system around our reference point.

We now want to disentangle the various components of this dynamics and for that purpose look at the points on an infinitesimal circle around the reference point, i.e. we look at the displacement vector that originally is



$$\xi(t_0) = r_0 \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

Let us first investigate the effect of a diagonal B. We take

$$B^A{}_C = \frac{1}{2}\theta\delta^A_C$$

so that $B^A{}_A = \theta$. We then have

$$\frac{\mathrm{d}\xi^A}{\mathrm{d}t} = \frac{1}{2}\theta\xi^A$$

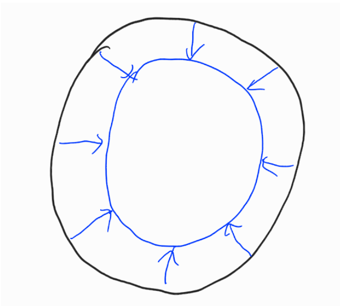
or

$$\dot{r} = \frac{1}{2}\theta r$$

We can also express this as a fractional change in area A

$$\theta = \frac{\dot{A}}{A}$$

which gives us the interpretation of θ as the relative expansion of the area.



Having taken care of the trace part we now investigate the traceless part. Let us first look at

$$B = \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix}$$

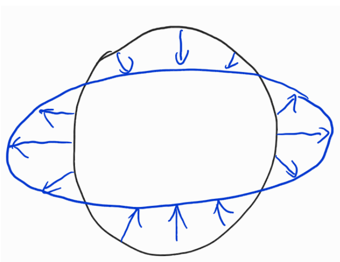
which is the traceless, symmetric part. This results in

$$\begin{aligned} \frac{\mathrm{d}\xi^A}{\mathrm{d}t} &= r_0 \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \\ &= r_0 \begin{pmatrix} \sigma_+ \cos\varphi + \sigma_\times \sin\varphi \\ \sigma_\times \cos\varphi - \sigma_+ \sin\varphi \end{pmatrix} \end{aligned}$$

which is an ellipse. Setting $\sigma_\times = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = r_0 \begin{pmatrix} \sigma_+ \cos\varphi \\ -\sigma_+ \sin\varphi \end{pmatrix} = \begin{pmatrix} \sigma_+ \xi^1 \\ -\sigma_+ \xi^2 \end{pmatrix}$$

This does not change the area but rather deform the circle into an ellipse with the axes aligned to the coordinate axes.



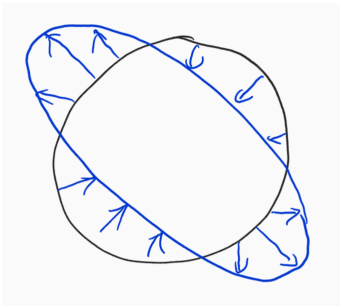
If on the other hand we set $\sigma_+ = 0$ we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = r_0 \begin{pmatrix} \sigma_\times \sin\varphi \\ \sigma_\times \cos\varphi \end{pmatrix} = \begin{pmatrix} \sigma_\times \xi^2 \\ \sigma_\times \xi^1 \end{pmatrix}$$

By taking the orthogonal linear combinations $\xi^u = \xi^1 + \xi^2$ and $\xi^v = \xi^1 - \xi^2$, we can write this as

$$\begin{aligned} \frac{\mathrm{d}\xi^u}{\mathrm{d}t} &= \sigma_\times \xi^u \\ \frac{\mathrm{d}\xi^v}{\mathrm{d}t} &= -\sigma_\times \xi^v \end{aligned}$$

which again is an area conserving elliptical deformation, but this time the axes of the ellipse are diagonal in the coordinate axes.



Finally we look at the antisymmetric part of B that we write as

$$B = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

From it we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = r_0 \begin{pmatrix} \omega \sin\varphi \\ -\omega \cos\varphi \end{pmatrix} = \begin{pmatrix} \omega \xi^2 \\ -\omega \xi^1 \end{pmatrix}$$

Writing this as

$$\begin{pmatrix} \mathrm{d}\xi^1 \\ \mathrm{d}\xi^2 \end{pmatrix} = \begin{pmatrix} \omega \mathrm{d}t \xi^2 \\ -\omega \mathrm{d}t \xi^1 \end{pmatrix}$$

exposes it as a rotation by an infinitesimal angle $\omega \mathrm{d}t$. The lessons learned from this example apply in general. We can always decompose a linear deformation into three parts:

- A trace part, corresponding to expansion
- A symmetric, traceless part corresponding to shear
- An antisymmetric part corresponding to rotation (or torsion)

1.11.2 Timelike geodesics

We now want to apply the general lessions from deformable media to congruences of timelike geodesics. We want to study how the deviation vector ξ^α behaves as a function of proper time. In order to study this let us first go into local flat coordinates at some reference point. In these coordinates, the eigentime τ is equal to the parameter time t and thus the tangent vector to the geodesic u^α may be written as

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \overset{*}{=} \frac{\mathrm{d}x^\alpha}{\mathrm{d}x^t} = \delta^\alpha_t$$

We can use this, still in local flat coordinates, to define a "subtracted metric"

$$h_{\alpha\beta} = u_\alpha u_\beta - g_{\alpha\beta} \stackrel{*}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

As we can see, $h_{\alpha\beta}$ is just the spatial (Euclidean) part of the metric in locally flat coordinates. But the definition $h_{\alpha\beta} = u_\alpha u_\beta - g_{\alpha\beta}$ is general. We thus conclude that $h_{\alpha\beta}$ is the spatial part of the metric $g_{\alpha\beta}$ in our geodesic congruence. Per construction it is transverse to the tangent vector u^α

$$\begin{aligned} h_{\alpha\beta} u^\beta &= (u_\alpha u_\beta - g_{\alpha\beta}) u^\beta \\ &= u_\alpha \underbrace{u_\beta u^\beta}_{1} - g_{\alpha\beta} u^\beta \\ &= u_\alpha - u_\alpha \\ &= 0 \end{aligned}$$

We now introduce the (suggestively named) tensor

$$B_{\alpha\beta} := u_{\alpha;\beta}$$

Because of the geodesic equation $B_{\alpha\beta} u^\beta = 0$. Also

$$\begin{aligned} u^\alpha B_{\alpha\beta} &= u^\alpha u_{\alpha;\beta} \\ &= \frac{1}{2} (\underbrace{u^\alpha u_\alpha}_{1})_{;\beta} \\ &= 0 \end{aligned}$$

so $B_{\alpha\beta}$ is transverse, too. Its full geometric significance is exposed by considering

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} \xi^\alpha &= \xi^\alpha{}_{;\beta} u^\beta \\ &= u^\alpha{}_{;\beta} \xi^\beta \\ &= B^\alpha{}_\beta \xi^\beta \end{aligned}$$

where we have used the vanishing lie derivative $\mathcal{L}_u \xi^\alpha = 0$. This relation has the same structure as (11) with the only difference being the number of dimensions. Our elasticity example was two dimensional, while $B_{\alpha\beta}$ is actually the deformation tensor of the 3-dimensional spatial cross section of the geodesic congruence. (Note that because of the orthogonality relations $B_{\alpha\beta} u^\beta = u^\beta B_{\beta\alpha} = 0$, $B_{\alpha\beta}$ has no time component.) We can thus decompose the deformation tensor into a trace part θ, pertinent to the expansion of the cross section, a traceless symmetric part σ and an antisymmetric part ω which encode shear and rotation as

$$B_{\alpha\beta} = \frac{1}{3} \theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

Per definition $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$, $\sigma^\alpha{}_\alpha = 0$ and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. Most importantly,

$$\theta = \frac{\dot{V}}{V}$$

so θ is the relative rate of change of the cross sectional volume of the congruence, in complete analogy to the cross sectional area of the deformable medium.

1.11.3 The Raychaudhuri equation and the focusing theorem

Let us now explore how the deformation tensor $B_{\alpha\beta}$ evolves with time. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}B_{\alpha\beta} &= B_{\alpha\beta;\mu}u^{\mu}\\ &= u_{\alpha;\beta\mu}u^{\mu}\\ &= (u_{\alpha;\mu\beta}-R_{\alpha\nu\beta\mu}u^{\nu})u^{\mu}\\ &= \underbrace{(u_{\alpha;\mu}u^{\mu})}_{0}{}_{;\beta}-u_{\alpha;\mu}u^{\mu}{}_{;\beta}-R_{\alpha\nu\beta\mu}u^{\nu}u^{\mu}\\ &= -B_{\alpha\mu}B^{\mu}{}_{\beta}-R_{\alpha\nu\beta\mu}u^{\nu}u^{\mu}\end{aligned}$$

We are specifically interested in the expansion part θ, so we trace this equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\theta &= \frac{\mathrm{d}}{\mathrm{d}t}B^{\alpha}{}_{\alpha}\\ &= -B^{\alpha}{}_{\mu}B^{\mu}{}_{\alpha}-R^{\alpha}{}_{\nu\alpha\mu}u^{\nu}u^{\mu}\\ &= -\left(\frac{1}{3}\theta\delta^{\alpha}_{\mu}+\sigma^{\alpha}{}_{\mu}+\omega^{\alpha}{}_{\mu}\right)\left(\frac{1}{3}\theta\delta^{\mu}_{\alpha}+\sigma^{\mu}{}_{\alpha}+\omega^{\mu}{}_{\alpha}\right)-R_{\nu\mu}u^{\nu}u^{\mu}\\ &= -\frac{1}{9}\theta^2\delta^{\alpha}_{\mu}\delta^{\mu}_{\alpha}-\sigma^{\alpha}{}_{\mu}\sigma^{\mu}{}_{\alpha}-\omega^{\alpha}{}_{\mu}\omega^{\mu}{}_{\alpha}-\frac{2}{3}\theta\underbrace{\sigma^{\alpha}{}_{\alpha}}_{0}-\frac{2}{3}\theta\underbrace{\omega^{\alpha}{}_{\alpha}}_{0}-2\underbrace{\sigma_{\alpha\mu}\omega^{\mu\alpha}}_{0}-R_{\nu\mu}u^{\nu}u^{\mu}\\ &= -\frac{1}{3}\theta^2-\sigma^{\alpha\beta}\sigma_{\alpha\beta}+\omega^{\alpha\beta}\omega_{\alpha\beta}-R_{\nu\mu}u^{\nu}u^{\mu}\end{aligned}$$

Note that all cross terms in the above derivation vanish, the first two because both σ and ω are traceless and the last one because σ is symmetric while ω is antisymmetric. What we are left with is the Raychaudhuri equation for timelike geodesic congruences

$$\dot{\theta}=-\frac{1}{3}\theta^2-\sigma^{\alpha\beta}\sigma_{\alpha\beta}+\omega^{\alpha\beta}\omega_{\alpha\beta}-R_{\nu\mu}u^{\nu}u^{\mu}$$

Remembering that θ is the relative rate of change of the cross sectional volume of the congruence, this equation yields the "acceleration" of volume change. When we look at the individual terms we notice that $\sigma^{\alpha\beta}\sigma_{\alpha\beta}\geqslant 0$ because it has purely spatial components. Similarly $\omega^{\alpha\beta}\omega_{\alpha\beta}\geqslant 0$. We can in fact choose a congruence so that it has no rotation. Such a congruence is called hypersurface orthogonal and its tangent vectors u_{μ} can be obtained from a scalar function Φ as $u_{\mu}\propto\Phi_{,\mu}$. (This is known as Frobenius' theorem and is the generalization of the statement that a curl free vector field can be written as the gradient of a scalar potential.) For such a congruence, θ will decrease over time unless the last term can give a positive contribution. Let us see what that implies. Using the field equations in the form(9), we can write

$$R_{\nu\mu}u^{\nu}u^{\mu}=8\pi\left(T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T\right)u^{\nu}u^{\mu}$$

In locally flat coordinates where $u^{\alpha}=\delta^{\alpha}_{t}$ and the diagonal elements of the energy momentum tensor $T_{tt}\overset{*}{=}\rho$ as well as $T_{ii}\overset{*}{=}p_i$ (no summantion implied) we have

$$\begin{aligned}R_{\nu\mu}u^{\nu}u^{\mu} &= 8\pi\left(T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T^{\mu}{}_{\mu}\right)u^{\nu}u^{\mu}\\ &= 8\pi\left(T_{tt}-\frac{1}{2}g_{tt}T^{\mu}{}_{\mu}\right)\\ &= 4\pi(T_{tt}-g_{tt}T^{i}{}_{i})\\ &\overset{*}{=} 4\pi(T_{tt}+T_{ii})\\ &= 4\pi\left(\rho+\sum_i p_i\right)\end{aligned}$$

We see that for "usual" matter this quantity is positive, so the last term in the Raychaudhuri equation is indeed negative, too. The condition

$$R_{\nu\mu}u^\nu u^\mu \geqslant 0$$

is known as the strong energy condition and assuming it, we conclude that the expansion coefficient of a hypersurface orthogonal geodesic congruence decreases

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - R_{\nu\mu}u^\nu u^\mu \leqslant 0$$

If we assume that both the second and the third term vanish $\sigma^{\alpha\beta}\sigma_{\alpha\beta} = R_{\nu\mu}u^\nu u^\mu = 0$, we obtain the case of minimal decrease of θ. Thus we always have an upper bound for $\dot{\theta}$

$$\dot{\theta} \leqslant -\frac{1}{3}\theta^2$$

which we can integrate to

$$\int_{\theta_0}^{\theta(\tau)} \frac{\mathrm{d}\theta}{\theta^2} \leqslant -\frac{1}{3}\int_0^\tau \mathrm{d}t$$

so

$$-\frac{1}{\theta(\tau)} + \frac{1}{\theta_0} \leqslant -\frac{1}{3}\tau$$

or

$$\theta(\tau) \leqslant \frac{1}{\frac{1}{\theta_0} + \frac{\tau}{3}}$$

If we start with a congruence that is initially converging, i.e. for which $\theta_0 < 0$, there is a strict upper bound for θ. Specifically, when $\tau \to -3/\theta_0$ the upper bound enforces $\theta(\tau) \to -\infty$. At the time $t = -3/\theta_0$ at the latest all geodesics in the congruence therefore converge onto one point, the caustic. This is statement is known as focusing theorem.

2 Classical black holes

We now turn our attention to a class of exact solutions of the Field equations that are generically referred to as black holes. The term black hole is so common these days, that it needs some work to disentangle the popular concept of a black hole from the actual object we are going to investigate. It is true that a black hole is a compact object from which, in a sense that we will make precise, not even light can escape. What is less clear from the popular concept is that classical black holes are *vacuum* solutions to the field equations - i.e. in the region where the black hole metric is defined, the energy momentum tensor vanishes. This might sound counter-intuitive when thinking about black holes as the end product of stellar collapse or similarly cataclysmic events. We will see however that in a certain sense the final product of a classical gravitational collapse can be the vanishing of all the matter that has collapsed out of our manifold and into a singularity, which we have to remove from our description of spacetime. Such singularities are then typically shielded from the outside world by horizons, which are of great interest by themselves.

2.1 The classical Schwarzschild black hole

Before continuing with general considerations, let us take a look at the classical Schwarzschild black hole. The Schwarzschild metric (which in the form usually used is actually due to Droste and Weyl) reads

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \frac{1}{1 - \frac{r_s}{r}}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 \tag{12}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$$

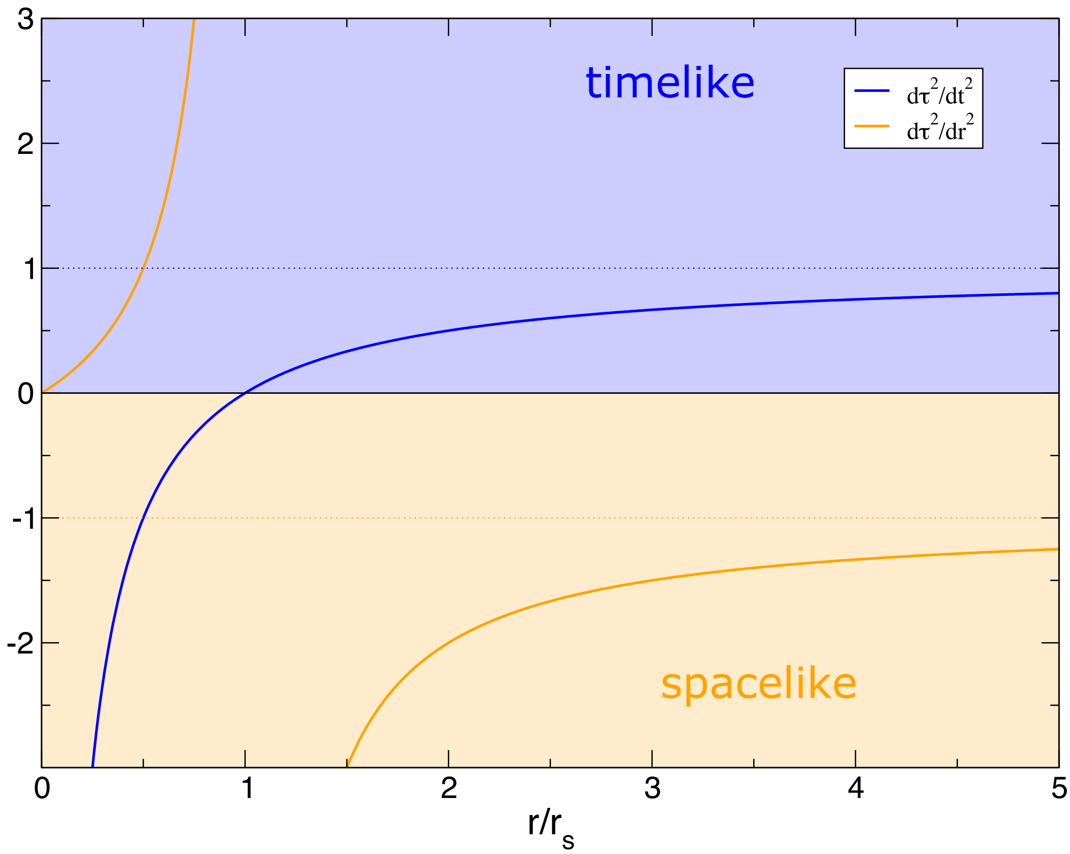
First we note that when $r \gg r_s$ (r_s is known as the Schwarzschild radius), the metric (12) goes over into flat Minkovski spacetime in spherical coordinates. We can therefore identify t with the eigentime of an asymptotic (i.e. $r \gg r_s$) observer and r with the radial coordinate that such an observer measures. It is interesting to note that one can make this identifications, but they are not unique. Every identification that differs by $O(r_s/r)$ will be equally valid. What is nice about this particular identification though is the fact that the surface of a sphere at r is always $4\pi r^2$ and the angular coordinates θ and φ will behave 'normally'.

2.1.1 Some basic properties of the Schwarzschild solution

The interesting behaviour of the Schwarzschild metric occurs in the t and r coordinates. Let us investigate this by taking two points at the same angular coordinates θ and φ as well as the same coordinate r, while being separated in the coordinate t by an infinitesimal amount $\mathrm{d}t$. For these two points

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 \tag{13}$$

Let us start with the asymptotic observer at $r \gg r_s$. For her, $\mathrm{d}\tau = \mathrm{d}t$. However when she starts approaching $r \to r_s$, $\mathrm{d}\tau < \mathrm{d}t$ until at $r = r_s$ finally $\mathrm{d}\tau = 0$. This means that the time between the two events shrinks, even though the interval $\mathrm{d}t$ between them stays constant. Since $\mathrm{d}t$ is the time interval as seen by an asymptotic observer, we may say that static clocks in the Schwarzschild metric tick slower if they are closer to the Schwarzschild radius r_s - but still outside of it. At r_s time freezes as seen from the outside observer, but what about $r < r_s$? In this region $\mathrm{d}\tau^2 < 0$ as one can see from the following figure:



At face value $\mathrm{d}\tau{=}0$ means that points are lightlike separated while $\mathrm{d}\tau^2 < 0$ signifies that they are spacelike separated. Thus at r_s t stops being a time coordinate and for $r < r_s$ it turns into a spatial coordinate. Something similar, but in reverse, happens to the coordinate r. If we look at two points at the same angular coordinates θ and φ as well as the same coordinate t that are separated in the coordinate r by $\mathrm{d}r$, we see that

$$\mathrm{d}\tau^2 = -\frac{1}{1-\frac{r_s}{r}}\mathrm{d}r^2$$

An asymptotic observer will see a distance $\mathrm{d}s = \sqrt{-\mathrm{d}\tau^2} \xrightarrow{r\to\infty} \mathrm{d}r$ between these two points. As he approaches r_s from outside, the distance $\mathrm{d}s = \sqrt{-\mathrm{d}\tau^2}$ between two points separated in coordinate r by $\mathrm{d}r$ will get larger and actually diverges at r_s (see figure). For $r < r_s$, the sign in $\mathrm{d}\tau^2$ flips and we see that the two points at the same t, θ, φ but a distance $\mathrm{d}r$ apart are now separated in a timelike manner. Thus at $r = r_s$, the coordinates t and r have exchanged places: t now is a spatial coordinate and r is the time coordinate.

While flipping a time coordinate into a spatial coordinate is odd, it does not raise any particular questions about the physical interpretation. In the reverse case this is not true, since for a time coordinate we need to know what is future and what is past. The Schwarzschild metric is of no help there since it only provides us with an expression for $\mathrm{d}\tau^2$ which has the two solutions

$$\mathrm{d}\tau = \pm\sqrt{\frac{r}{r_s - r}}\mathrm{d}r$$

and we need to invoke some physical reasoning. We can e.g. picture some observer that is infalling towards $r = 0$ from the outside. If we ignore all divergences and complications that might appear along the way, we can just identify the movement forward in time with that inward, i.e. with $\mathrm{d}r < 0$. It would thus be natural for objects that move inward to identify forward in time with a negative $\mathrm{d}r$. Similarly it would be natural to identify positive $\mathrm{d}r$ with *outmoving* objects. This is the point where we are duly reminded that the Schwarzschild solution in a *vacuum* solution of the field equations and not what we might intuitively associate with a black hole. It does not distinguish between $\pm\mathrm{d}\tau$ and thus the time reversal of a geodesic is also a geodesic - a point which we will come back to in great detail.

There is one more point that we need to be aware of: While r is a time coordinate inside r_s, it still determines the size of the spherical shell on which the angular coordinates live to be $4\pi r^2$. The spatial volume thus changes with time r and in particular it becomes zero at $r = 0$. You may remember that $r = 0$ was not part of the Schwarzschild solution to begin with - it had to be excluded as some components of the curvature tensor diverged there. This is one way of how a singularity manifests itself in general relativity.

2.1.2 A free falling radial observer in the Schwarzschild metric

Let us now try to understand the fate of a radially free falling observer in a bit more detail. We construct the geodesic equation

$$\dot{u}^\alpha + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0$$

with

$$u^t = \dot{t} \quad u^r = \dot{r} \quad u^\theta = u^\varphi = 0$$

where the overdot corresponds to a derivative with respect to the eigentime τ. The only two nontrivial components of the geodesic equation are thus

$$\ddot{t} + 2\Gamma^t_{rt}\dot{r}\dot{t} = 0$$

and

$$\ddot{r} + \Gamma^r_{tt}\dot{t}\dot{t} + \Gamma^r_{rr}\dot{r}\dot{r} = 0$$

Plugging in the Christoffel symbols we find

$$\begin{aligned} \ddot{t} - \frac{r_s}{r(r_s - r)}\dot{r}\dot{t} &= 0 \\ \ddot{r} - \frac{r_s(r_s - r)}{2r^3}\dot{t}^2 + \frac{r_s}{2r(r_s - r)}\dot{r}^2 &= 0 \end{aligned} \tag{14}$$

The first of these two equations may be rewritten as

$$\begin{aligned} 0 &= \frac{r_s - r}{r}\ddot{t} - \frac{r_s}{r^2}\dot{r}\dot{t} \\ &= \Big(\frac{r_s}{r} - 1\Big)\ddot{t} - \frac{r_s}{r^2}\dot{r}\dot{t} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\Big(\frac{r_s}{r} - 1\Big)\dot{t}\Big) \end{aligned}$$

So the bracketed quantity (with, per convention, a negative sign)

$$H = \Big(1 - \frac{r_s}{r}\Big)\dot{t} \tag{15}$$

is a constant of motion. Plugging this into the second equation in (14) we find

$$\begin{aligned} 0 &= \ddot{r} - \frac{r_s(r_s - r)}{2r^3}\dot{t}^2 + \frac{r_s}{2r(r_s - r)}\dot{r}^2 \\ &= \ddot{r} - \frac{r_s\big(\frac{r_s}{r} - 1\big)}{2r^2}\dot{t}^2 + \frac{r_s}{2r^2\big(\frac{r_s}{r} - 1\big)}\dot{r}^2 \\ &= \ddot{r} - \frac{r_s}{2r^2\big(\frac{r_s}{r} - 1\big)}H^2 + \frac{r_s}{2r^2\big(\frac{r_s}{r} - 1\big)}\dot{r}^2 \\ &= \ddot{r} + \frac{r_s}{2r(r_s - r)}(\dot{r}^2 - H^2) \end{aligned}$$

Multiplying by $2\dot{r}r/(r_s - r)$ this becomes

$$\begin{aligned} 0 &= 2\dot{r}\ddot{r}\frac{1}{r_s - r} + \frac{\dot{r}r_s}{(r_s - r)^2}(\dot{r}^2 - H^2) \\ &= 2\dot{r}\ddot{r}\frac{r}{r_s - r} + \frac{\dot{r}(r_s - r) - (-r)\dot{r}}{(r_s - r)^2}(\dot{r}^2 - H^2) \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}(\dot{r}^2)\frac{r}{r_s - r} + \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg(\frac{r}{r_s - r}\bigg)(\dot{r}^2 - H^2) \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg(\frac{r}{r_s - r}(\dot{r}^2 - H^2)\bigg) \end{aligned}$$

which gives us another constant of motion

$$K = \frac{r}{r_s - r}(\dot{r}^2 - H^2) \tag{16}$$

Let us now assume that we start the free fall at a radius $r_0 > r_s$ with an initial velocity $\dot{r} = 0$. This results in

$$K = H^2\frac{r_0}{r_0 - r_s} \tag{17}$$

From (15) we find

$$H = \Big(1 - \frac{r_s}{r_0}\Big)\dot{t}_0$$

Since at r_0 we have $\mathrm{d}r = 0$, it follows that

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r_0}\right)\mathrm{d}t^2$$

so

$$\dot{t}_0 = \left.\frac{\mathrm{d}t}{\mathrm{d}\tau}\right|_{r_0} = \frac{1}{\sqrt{1 - \frac{r_s}{r_0}}}$$

and thus

$$H = \sqrt{1 - \frac{r_s}{r_0}}$$

Plugging this into (17) we finally obtain

$$K = 1$$

Thus the condition (16) reads

$$1 = \frac{r}{r_s - r}\left(\dot{r}^2 - \frac{r_0 - r_s}{r_0}\right)$$

or

$$\dot{r} = \pm\sqrt{\frac{r_s}{r} - \frac{r_s}{r_0}}$$

For a radially infalling observer the negative sign is appropriate, so

$$\mathrm{d}\tau = -\frac{\mathrm{d}r}{\sqrt{\frac{r_s}{r} - \frac{r_s}{r_0}}} \tag{18}$$

Let us now suppose that some observer falls into the black hole, starting from rest at r_0. Integrating the above equation will yield the time elapsed for the observer until it reaches $r = 0$. We obtain

$$\begin{aligned}
\tau &= -\int_{r_0}^{0} \frac{\mathrm{d}r}{\sqrt{\frac{r_s}{r} - \frac{r_s}{r_0}}} \\
&= \frac{1}{\sqrt{\frac{r_s}{r_0}}}\int_0^{r_0} \frac{\mathrm{d}r}{\sqrt{\frac{r_0}{r} - 1}} \\
&= \left\{\begin{array}{rcl} \frac{r}{r_0} & = & \sin^2 y \\ \mathrm{d}r & = & r_0\, 2\sin y\cos y\,\mathrm{d}y \\ r = 0 & \to & y = 0 \\ r = r_0 & \to & y = \frac{\pi}{2} \end{array}\right\} \\
&= \sqrt{\frac{r_0}{r_s}}\int_0^{\frac{\pi}{2}} \frac{r_0\, 2\sin y\cos y\,\mathrm{d}y}{\sqrt{\frac{1}{\sin^2 y} - 1}} \\
&= 2\sqrt{\frac{r_0^3}{r_s}}\int_0^{\frac{\pi}{2}} \frac{\sin y\cos y\,\mathrm{d}y}{\sqrt{\frac{\cos^2 y}{\sin^2 y}}} \\
&= 2\sqrt{\frac{r_0^3}{r_s}}\int_0^{\frac{\pi}{2}} \sin^2 y\,\mathrm{d}y \\
&= \frac{\pi}{2}\sqrt{\frac{r_0^3}{r_s}}
\end{aligned} \tag{19}$$

which is finite. What might be even more surprising is the fact that nothing special at all seems to happen at r_s. In fact, we can even drop our initial requirement that $r_0 > r_s$ if we are not afraid of the integration constant H becoming imaginary. In any case we see, that an observer which is radially free falling into the black hole will reach its central singularity in a finite time.

But how does this go together with our previous observation that near r_s static clocks tick slower as seen by a far away observer? To answer this question, let us look at what an asymptotic observer actually sees. For her, the relevant time is not τ but t. The connection between the two is given by (15), so we have

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}\tau} &= \frac{H}{1-\frac{r_s}{r}} \\ &= \frac{\sqrt{1-\frac{r_s}{r_0}}}{1-\frac{r_s}{r}}\end{aligned}$$

Using (18) we obtain

$$\frac{1-\frac{r_s}{r}}{\sqrt{1-\frac{r_s}{r_0}}}\mathrm{d}t = -\frac{\mathrm{d}r}{\sqrt{\frac{r_s}{r}-\frac{r_s}{r_0}}}$$

or

$$\begin{aligned}\mathrm{d}t &= -\sqrt{\frac{r_0}{r_s}}\frac{\sqrt{1-\frac{r_s}{r_0}}}{\sqrt{\frac{r_0}{r}-1}\left(1-\frac{r_s}{r}\right)}\mathrm{d}r \\ &= -\frac{\sqrt{\frac{r_0}{r_s}-1}}{\sqrt{\frac{r_0}{r}-1}\left(1-\frac{r_s}{r}\right)}\mathrm{d}r\end{aligned}$$

Integrating this equation, we can immediately see that the integrand on the right hand side has a pole $\propto -(r-r_s)^{-1}$ at $r = r_s$. Integrated, this will lead to a logarithmic divergence

$$t \propto -\ln(r-r_s) \xrightarrow{r\to r_s} \infty$$

Consequently, an asymptotic observer will see that an infalling object will take an infinite time until it reaches even r_s.

Finally, let us look at an observer which is not asymptotic but hovers at some fixed radius $r_1 > r_s$. Its eigentime is given by (13), so there will be a fixed factor $\sqrt{1-\frac{r_s}{r_1}}$ between its observed time and that of an asymptotic observer. So this observer will see the radial infall proceeding faster but still come to a halt as $r \to r_s$.

2.1.3 The horizon

We have now investigated the Schwarzschild metric enough to clearly discern two interesting regions: The first one is around $r = 0$, where time ends for an infallig observer. Since we had to exclude $r = 0$ from our spacetime to obtain the Schwarzschild solution in the first place, it is rather evident that we have a singularity there. Singularities appearing in a physical theory generically show the limit of applicability of that theory. In case of general relativity, that limit is reached at $r = 0$ and one can speculate whether quantum effects or some other mechanism ultimately correct GR in such a way as to remove that singularity. The other interesting region of the Schwarzschild solution is around $r = r_s$. When looking at the metric $r = r_s$ sure looks like a singularity, too. And we have just computed that an observer outside r_s never sees a free falling object cross r_s. But on the other hand, the free falling observer reaches $r = 0$ in a finite time and there was no problem integrating its trajectory across r_s. So what is going on there?

To get a first idea of what is happening at r_s, let us imagine two observers that communicate with each other (like a brave explorer going near the black hole and her friend waiting at a safe distance). They would like to communicate efficiently, so they decide to locate themselves at the same angular coordinates θ and φ and use radio communication (obviously, any other choice would make their communication slower or less efficient). Since the communication is achieved with light, $\mathrm{d}\tau = 0$ for the message. In addition, $\mathrm{d}\theta = \mathrm{d}\varphi = 0$ since the observers have sensibly chosen to be at the same angular coordinates. The message the explorer sends thus travels radially as

$$0 = \mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \frac{1}{1 - \frac{r_s}{r}}\,\mathrm{d}r^2$$

so

$$\mathrm{d}r = \pm\left(1 - \frac{r_s}{r}\right)\mathrm{d}t$$

Initially, when $r > r_s$, the two choices offered by the sign are of course a message that is sent radially inward (the negative sign) and a message that is sent radially outward (the positive sign). Obviously the correct one (the one that the observer at a safe distance will receive) is the radially outward one so we have

$$\mathrm{d}r = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t \tag{20}$$

But once the explorer crosses r_s so that $r_s > r$, the factor on the right hand side becomes negative and $\mathrm{d}r$ will actually decrease *even for an outgoing light ray*. The messages from the explorer will no more reach the distant observer. But, one might ask, what about the radially ingoing light rays? Their sign flips, too, so for them we have

$$\mathrm{d}r = \left(\frac{r_s}{r} - 1\right)\mathrm{d}t \tag{21}$$

and the factor is positive. So indeed if $\mathrm{d}t$ was positive for them, $\mathrm{d}r$ would be positive, too and the light ray would travel outward. But remember that t is no more the time coordinate inside r_s, but rather a spatial coordinate. The time coordinate inside r_s is r and we already established that for a black hole $\mathrm{d}r < 0$ so time ends at $r = 0$. The correct way to read (20) and (21) is thus to conclude that for $r < r_s$ the "outgoing" light ray has a positive $\mathrm{d}t$ while the "ingoing" one has a negative one. None of these light rays (nor anything else from the explorer) will be able to reach the distant observer once $r < r_s$ because $\mathrm{d}r < 0$ is the flow of time for $r < r_s$ and nothing can escape that. The only way we could change that was to assume that the flow of time is in the other direction, i.e. $\mathrm{d}r > 0$ for $r < r_s$. But in that case it is not just the radio signals or light rays (both the "ingoing" and the "outgoing") that would reach the outside but literally everything inside r_s. It would actually do so in a finite proper time, which we have already calculated in (19) as the free fall time from $r_0 = r_s$

$$\tau = \frac{\pi}{2} r_s$$

The region $r = r_s$ thus is not a singularity but rather a strange feature in the causal structure of spacetime. Loosely speaking we may say that what is inside (i.e. $r < r_s$) either stays inside and can never influence again what happens outside (this is called a black hole) or, alternatively, everything inside will get expelled and nothing con go inside (this is called a white hole). In both cases the region $r = r_s$ acts like a one-sided membrane that allows objects to cross in one way but not the other. Such a structure is called an event horizon.

2.1.4 Near horizon coordinates

The region around r_s is certainly an interesting one and we would like to study it a bit more. For this purpose, we introduce the first of many coordinate transformations to follow: we will replace the radial coordinate r by the proper distance above the horizon ρ. We yet have to establish the connection between r and ρ, but we know that we can write the metric as

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r(\rho)}\right)\mathrm{d}t^2 - \mathrm{d}\rho^2 - r^2(\rho)\mathrm{d}\Omega^2$$

Since this is the Schwarzschild metric in other coordinates, we evidently have

$$\mathrm{d}\rho = \frac{\mathrm{d}r}{\sqrt{1 - \frac{r_s}{r}}}$$

We can integrate both sides of the equation to obtain

$$\rho = r\sqrt{1 - \frac{r_s}{r}} + r_s \mathrm{atan}\sqrt{1 - \frac{r_s}{r}}$$

where we have set the integration constant such that $\rho = 0$ at $r = r_s$, i.e. ρ is the proper distance from the horizon. In the vicinity of the horizon, i.e. for $r - r_s \ll r$, we can approximate

$$\rho \simeq 2r_s\sqrt{1 - \frac{r_s}{r}}$$

or

$$r \simeq \frac{r_s}{1 - \frac{\rho^2}{4r_s^2}}$$

Very close to (but outside) the horizon we can thus write the metric as

$$\mathrm{d}\tau^2 \simeq \left(\frac{\rho^2}{4r_s^2}\right)\mathrm{d}t^2 - \mathrm{d}\rho^2 - r^2(\rho)\mathrm{d}\Omega^2$$

We can define a rescaled time coordinate

$$\alpha = \frac{t}{2r_s}$$

with which the metric reads

$$\mathrm{d}\tau^2 \simeq \rho^2\mathrm{d}\alpha^2 - \mathrm{d}\rho^2 - r^2(\rho)\mathrm{d}\Omega^2$$

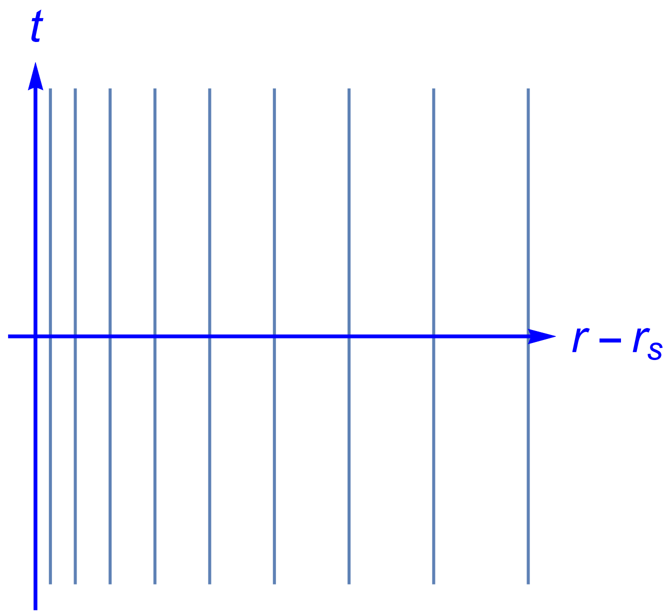
Interestingly, the α, ρ part of the metric looks exactly like two dimensional Minkovski-space in Rindler coordinates (see exercise sheet 6). An observer hovering at a fixed ρ above the horizon is thus locally equivalent to a Rindler observer at a fixed ρ. We remember that the Rindler observer at fixed ρ is constantly accelerating (in flat space) with an acceleration $a = \sqrt{-a^\mu a_\mu} = 1/\rho$. This is a nice example of the equivalence principle: a local observer can not distinguish between a gravitational field and acceleration. But we also see that as $\rho \to 0$ the equivalent acceleration diverges and, as we have seen in the exercise sheet, the Rindler trajectory turns from timelike to lightlike. This is a clear reminder that at some point "hovering a fixed distance above the horizon" becomes physically untenable. Since our coordinate system is based on these kind of observers (a fixed coordinate r implies a fixed ρ), it is no wonder that it breaks down near the horizon and we are encouraged to find a better coordinate system. It is rather evident how to do this locally. We can just replace the Rindler coordinates with carthesian ones. From the perspective of the local observer this will remove the constant acceleration of the coordinates and, by the equivalence principle, it will thus remove the gravitational acceleration. The resulting coordinates are thus free falling. We write the coordinate transformation as

$$\begin{aligned} T &= \rho \sinh \alpha \\ X &= \rho \cosh \alpha \end{aligned}$$

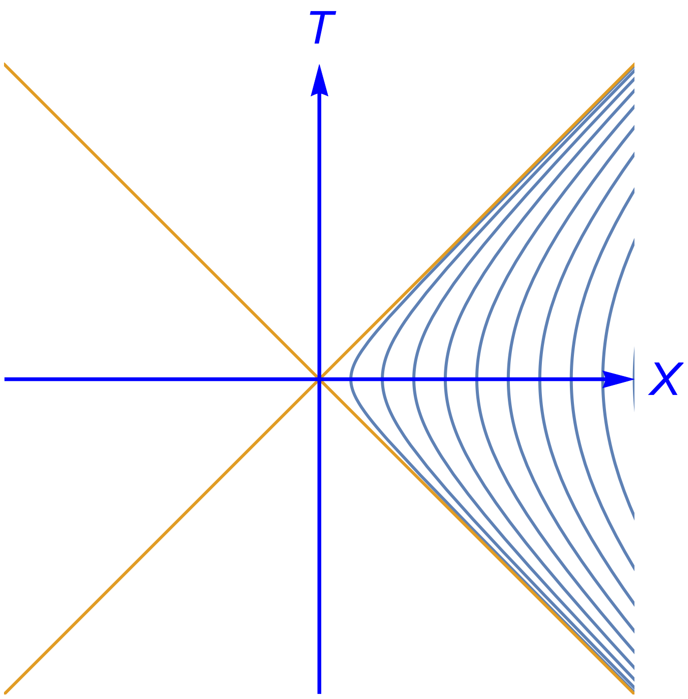
so the metric takes the form

$$\mathrm{d}\tau^2 \simeq \mathrm{d}T^2 - \mathrm{d}X^2 - r^2(\rho)\mathrm{d}\Omega^2$$

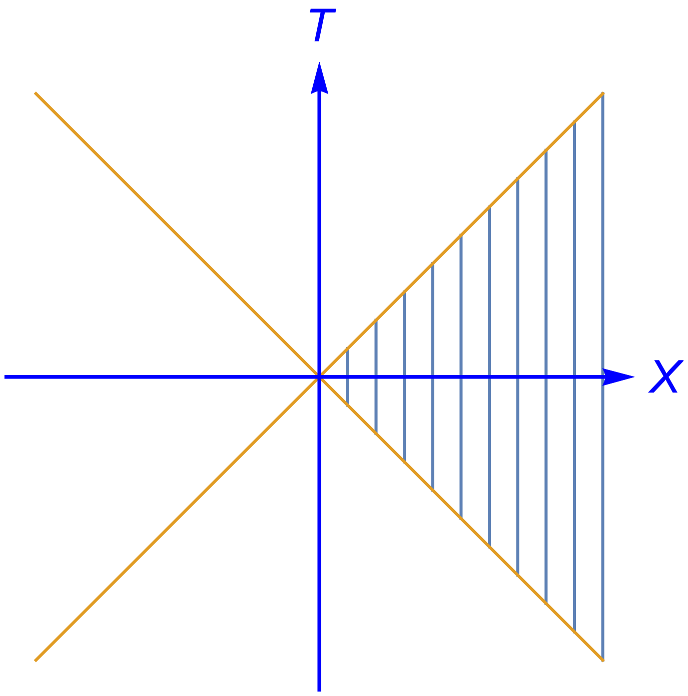
At this point it is helpful to actually draw the world lines of the different observers in the various coordinate systems. Let us start with observers that are hovering above the horizon at a fixed proper distance ρ. In Schwarzschild coordinates they look rather inconspicuous:



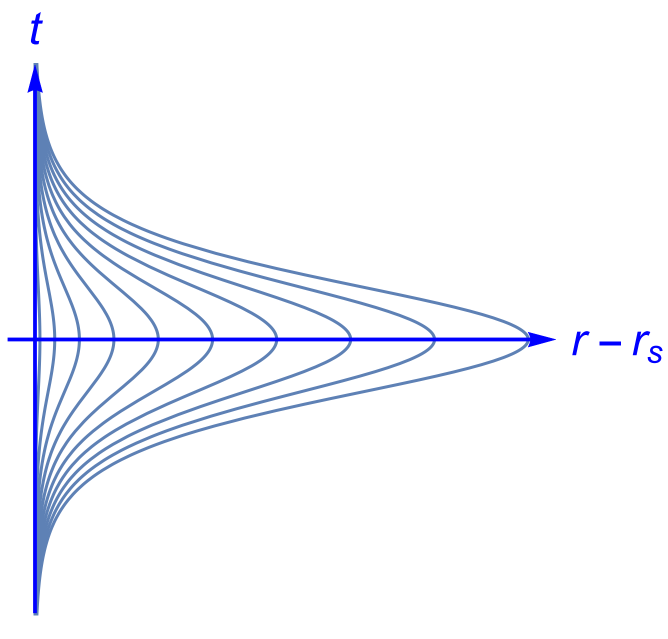
When we draw the same observers in freely falling near horizon coordinates, the picture looks rather different:



In these coordinates, freely falling observers seem much more natural:



And this is how their world lines look like in the original Schwarzschild coordinates:



Note especially how the world lines of freely falling observers stretch out to temporal infinity while they remain at $r > r_s$ in Schwarzschild coordinates. These same worldlines, in locally free falling coordinates, only run for a finite time, between between $T = \mp X$. It is this infinite stretch at the horizon that gives us the impression of some divergence there, but this is just due to our coordinate system.

2.1.5 The Kruskal extension of the Schwarzschild metric

Let us now try to address the inadequacies of the Schwarzschild coordinate system around r_s. Our general strategy will be to try repeating the construction of freely falling near horizon coordinates but in an exact and global manner. Looking at the Schwarzschild metric (12), the biggest obstacle seems to be that the radial and time coordinates do not scale equally as r changes. We can address this in a brute force way by introducing a new radial coordinate r^*, whose defining feature is that it changes in the same way as the time coordinate, i.e.

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r(r^*)}\right)(\mathrm{d}t^2 - \mathrm{d}r^{*2}) - r^2\,(r^*)\mathrm{d}\Omega^2$$

The relation between r and r^* is thus given by

$$\frac{1}{1 - \frac{r_s}{r}}\,\mathrm{d}r^2 = 1 - \frac{r_s}{r}\mathrm{d}r^{*2}$$

so

$$\frac{\mathrm{d}r}{1 - \frac{r_s}{r}} = \pm\mathrm{d}r^*$$

We obviously are interested in the positive solution, so

$$\frac{r}{r - r_s}\mathrm{d}r = \mathrm{d}r^*$$

or

$$\left(1 + \frac{r_s}{r - r_s}\right)\mathrm{d}r = \mathrm{d}r^*$$

which integrates to

$$\begin{aligned} r^* &= \int \left(1 + \frac{r_s}{r - r_s}\right)\mathrm{d}r \\ &= r + r_s\ln(r - r_s) + c \end{aligned}$$

To avoid a dimensionful argument of the logarithm, we choose $c = -r_s\ln(r_s)$ so that finally

$$r^* = r + r_s\ln\left(\frac{r}{r_s} - 1\right)$$

Note that when $r \to r_s$ then $r^* \to -\infty$, which is just a reflection of the fact that viewed in an asymptotic observer's time t, it takes infinitely long to reach r_s. These coordinates are therefore often referred to as tortoise coordinates (as in Achilles and the tortoise).

The radial-temporal part of our metric is now “Minkovski-like” except for a scaling factor. In exercise sheet 6 we have seen that light cone coordinates are very useful when talking about horizons (which are lightlike) and we will transition to them now. We define

$$u = t - r^* \qquad v = t + r^*$$

so that our metric becomes

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r(r^*)}\right)\mathrm{d}u\,\mathrm{d}v - r^2\,(r^*)\mathrm{d}\Omega^2$$

In these coordinates, the horizon $r = r_s$ is still removed to infinity. But being lightlike, we have a chance to bring it back to a finite value using a simple transformation of the light cone coordinates. The removal of the horizon to infinity was caused by a logarithm, so we can try curing it by exponentiating and introduce the new coordinates

$$U = -e^{-\frac{u}{2r_s}} \qquad V = e^{\frac{v}{2r_s}} \tag{22}$$

The signs in the exponent are dictated by the contribution of r^* to u (negative) and v (positive), the signs of U and V themselves are by convention. For dimensional reasons, u and v in the exponent needed to be divided by a length scale, with r_s being an obvious choice and the factor 2 being convenient as it will turn out. The total differentials are

$$\mathrm{d}U = -\frac{1}{2r_s} U \,\mathrm{d}u \qquad \mathrm{d}V = \frac{1}{2r_s} V \,\mathrm{d}v$$

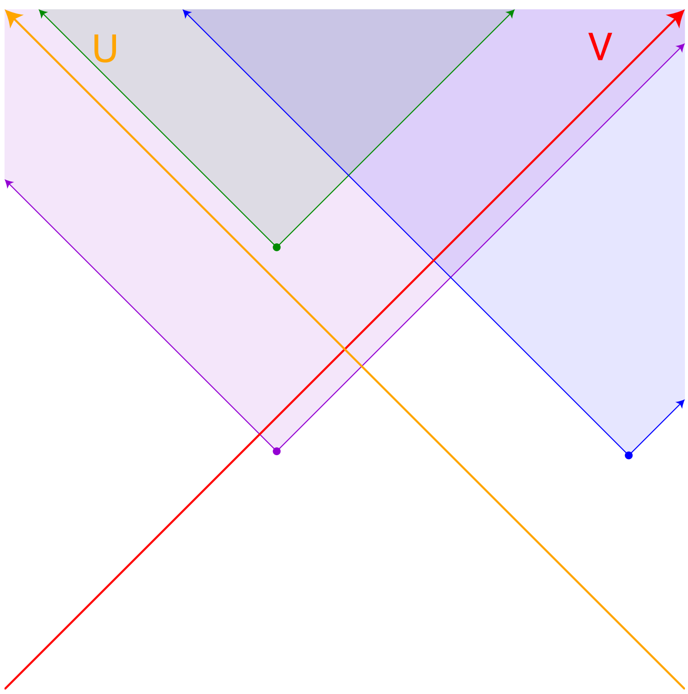
so

$$\mathrm{d}u = -2r_s \frac{\mathrm{d}U}{U} \qquad \mathrm{d}v = 2r_s \frac{\mathrm{d}V}{V}$$

The metric thus reads

$$\mathrm{d}\tau^2 = -\frac{4r_s^2}{UV}\left(1 - \frac{r_s}{r}\right)\mathrm{d}U\,\mathrm{d}V - r^2\,\mathrm{d}\Omega^2 \tag{23}$$

Note that both U and V are still light cone coordinates and that we can still trace light rays by setting one of them, and the angular coordinates, to a constant. This property will turn out to be very useful for elucidating the causal structure of the metric, as is evident in the following plot, which shows three events and their respective causal future:



It is customary to depict light cone coordinates as diagonal, as we have done here with U and V. The shaded areas emanating from the three events show all events that lie in its causal future, which is the region that can be reached from a point with paths that are timelike or lightlike and lie to the future of it. (Obviously we have ignored the angular coordinates here, but this extension is trivial). In our new coordinates, which are usually referred to as Kruskal or Kruskal-Szekeres coordinates, the future of an event is simply a wedge which has diagonals as boundary.

Let us now examine if we succeeded in bringing back the horizon to a finite value of our coordinates. Taking the product

$$UV = -e^{\frac{v-u}{2r_s}} = -e^{\frac{r^*}{r_s}} = -e^{\frac{r + r_s \ln\left(\frac{r}{r_s} - 1\right)}{r_s}} = -\left(\frac{r}{r_s} - 1\right)e^{\frac{r}{r_s}} = \left(1 - \frac{r}{r_s}\right)e^{\frac{r}{r_s}} \tag{24}$$

and defining an auxiliary variable $y = 1 - r/r_s$, we can write

$$UV = ye^{1-y}$$

or

$$-\frac{UV}{e} = -ye^{-y} \tag{25}$$

This is a transcendental equation of the form

$$x = we^w \tag{26}$$

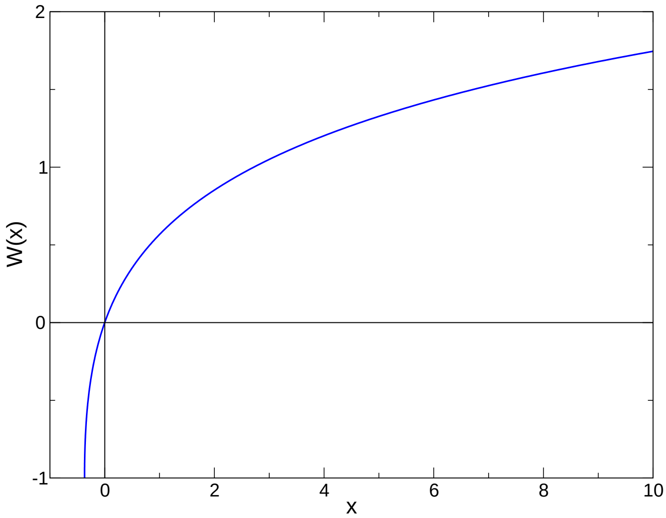
and ists solution with respect to w

$$w = W(x)$$

is known as the Lambert W-function or product logarithm. We can infer its essential properties from the inverse function (26). These are, specifically

- $W(0) = 0$
- $W(x) \gtrless 0$ for $x \gtrless 0$
- $\mathrm{d}x = \mathrm{d}w(1+w)e^w$, so $\frac{\mathrm{d}w}{\mathrm{d}x} = 0$ for $w = -1$

From the last property it follows that $W(x)$ has two branches, the relevant one being the upper branch $W_0(x)$ with $W_0(x) \geqslant -1$. Thus W_0 is defined in the range $[-1/e, \infty)$ and looks like this:



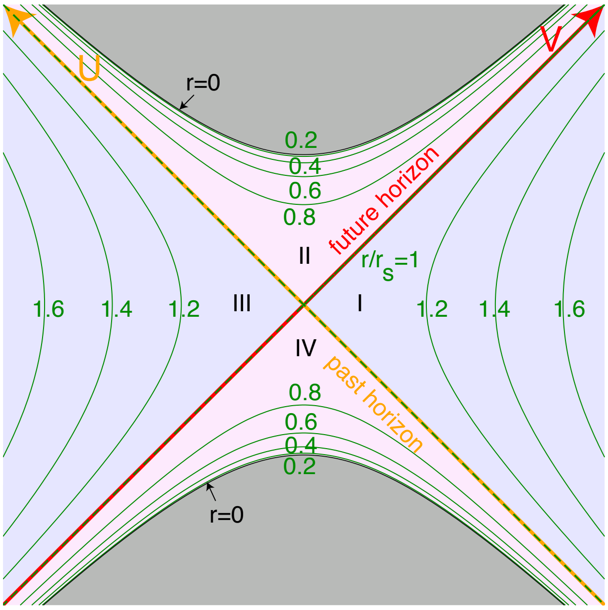
We now solve (25) and obtain

$$-y = W_0\left(-\frac{UV}{e}\right)$$

and substituting again $y = 1 - r/r_s$, we finally arrive at

$$r = r_s\left(1 + W_0\left(-\frac{UV}{e}\right)\right) \tag{27}$$

When we draw lines of constant r/r_s, they look like this:

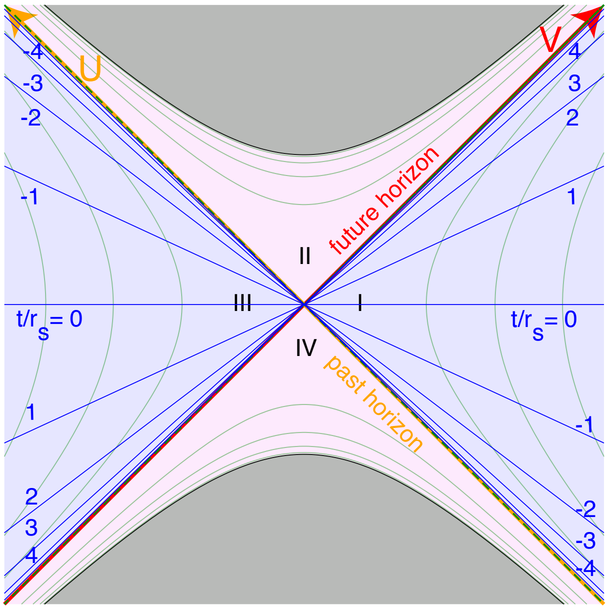


A few interesting features become rather obvious from this diagram. As we can see from (24), a constant r/r_s implies a constant UV. In coordinates U and V therefore all curves at constant r are hyperbolae. At $r = r_s$, the sign of UV flips, which causes the lines of constant r to flip from being timelike at $r > r_s$ to being spacelike at $r < r_s$ with the intermediate lightlike $r = r_s$. The minimal value possible for UV is -1, which corresponds to the hyperbola labeled $r = 0$. The grey shaded areas outside are thus not part of spacetime. The right, blue shaded wedge that is labeled I can be identified with the region outside the horizon $r > r_s$. Thinking about the causal future of points in region I, one can easily see that they will lie entirely in regions I and II only. We therefore have a (past) event horizon along the diagonal that is the U coordinate axis. Similarly, one can see that events in region I can only be influenced by events in either region I or IV. We therefore have a future event horizon along the diagonal that is the V coordinate axis. It is the upper half of this horizon (the V-axis for $V > 0$) which is the black hole horizon that we have considered so far. It allows crossing into the region II from region I, but not the other way. Region II is the inside of the black hole where we can see from the diagram that going forward in time is equivalent to going to smaller r until eventually one ends up at the singularity at $r = 0$. Consequently, by time reversal, region IV is the inside of the white hole. There, time starts at the singularity $r = 0$ and progresses as r increases towards $r = r_s$, where world lines can continue into the “outside” region I. In the other direction, the past horizon prohibits anything from going from the “outside” region I into the white hole region IV.

We thus seem to have covered everything we know about by regions I (outside), II (inside the BH) and IV (inside the WH), so what about region III? Geometrically, it is the region where the second branch of the hyperbolae $r > r_s$ lie, so it seems to be another “outside” region. There is however a difference to region I, which we can uncover by looking at the ratio

$$\frac{U}{V} = -e^{-\frac{u+v}{2r_s}} = -e^{-\frac{t}{r_s}}$$

This allows us drawing lines of equal asymptotic observer time t into the diagram above:



We see that in region III the forward time direction is given by $-\mathrm{d}t$, so in this sense region III is the time inverse of the outside region I. Note however that regions I and I are completely causally disconnected and thus can never be reached from each other. Physically it is thus completely unclear whether region III has any relevance. All we really know is that it appears in the extension of the Schwarzschild metric that we have just performed by going to U-V coordinates.

Finally, let us come back to the definition of the coordinates U and V (22) and note that both U and V are invariant under an *imaginary* shift in the coordinate $t \to t + 4\pi i r_s$. Another way to express this is that the metric (23) is *periodic* in imaginary time with a period $4\pi r_s$. At first this might seem like an odd curiosity, but in fact it is a crucial observation when coupling any quantum field theory to classical gravity. In a quantum field theory, periodicity in imaginary time is equivalent to temperature. Thus when we formulate a quantum field theory in Kruskal-Szekeres coordinates (and thus, ultimately, in the Schwarzschild metric), it will have a finite temperature

$$T = \frac{1}{4\pi r_s}$$

which is of course the famous Hawking temperature.

2.2 Penrose diagrams

We have seen in the previous section that a graphical representation is sometimes very useful in elucidating the causal structure of spacetime. In particular, we found Kruskal-Szekeres coordinates very helpful for understanding the causal structure of the Schwarzschild metric. One particularly useful feature was the fact that radial null geodesics were diagonal lines and thus the future light cone could easily be constructed. It was also quite helpful that the horizons, where the coordinate time t diverges, were at a finite position. The radial infall of an observer through the horizon could thus be graphically represented and one could read off how it looks like for both the asymptotic and the infalling observers. We can take this one step further and also move spatial and temporal infinities to a finite distance so they can be displayed conveniently. If we keep the radial null geodesics diagonal in this construction, we can carry over the important feature that light cones are easily drawn and the causal structure of a theory is clearly visible. Such diagrams are known as Penrose (or Penrose-Carter) diagrams. For radially symmetric metrics we can start with light cone coordinates u and v and construct

$$y^- = F(u) \qquad y^+ = F(v)$$

where the function $F(x)$ is chosen so that it maps real numbers into a finite interval. We then find

$$\mathrm{d}y^- = F'(u)\,\mathrm{d}u \qquad \mathrm{d}y^+ = F'(v)\,\mathrm{d}v$$

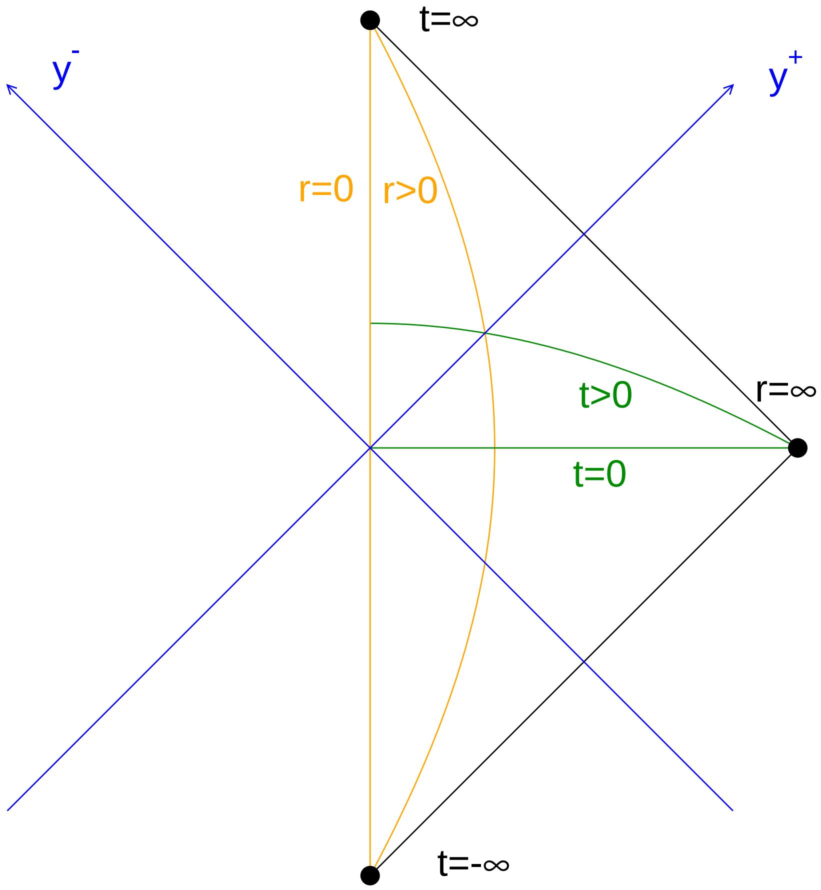
and thus if $\mathrm{d}y^\mp = 0$ it follows that $\mathrm{d}u = 0$ resp. $\mathrm{d}v = 0$. Therefore $y^\pm$ are also lightlike coordinates. We can e.g. take

$$F(x) = \tanh(x)$$

which will map $\mathbb{R} \to (-1, 1)$. As a simple example we may take flat Minkovski space in spherical coordinates, which has a metric

$$\mathrm{d}\tau^2 = \mathrm{d}u\mathrm{d}v - \mathrm{d}\Omega^2$$

where $u = t - r$ and $v = t + r$. Its Penrose diagram is a triangle:



We see that radial and temporal infinities have been compressed to a point while lightlike infinity (represented by the black diagonal lines connecting them) is a line in this representation. The left edge of the diagram, the vertical orange line, corresponds to $r = 0$ and thus signifies the end of spacetime in the radial direction.

We now apply this procedure to the Kruskal extension of the Schwarzschild metric. Here we make the choice

$$y^- = \mathrm{atan}(U) \qquad y^+ = \mathrm{atan}(V)$$

Thus if U and V would span all possible real values, we would have $y^{\pm} \in (-\pi/2, \pi/2)$. For $UV < 0$ (i.e. in the "outside" regions I and III) this is in fact the case. In the "inside" regions II and IV however, we have seen that spacetime stops at the singularity $r = 0$, which corresponds to $UV = 1$ as we have seen. In our new coordinates this corresponds to

$$\begin{aligned}
1 &= UV \\
&= \tan(y^-)\tan(y^+) \\
&= -\frac{(e^{iy^-} - e^{-iy^-})(e^{iy^+} - e^{-iy^+})}{(e^{iy^-} + e^{-iy^-})(e^{iy^+} + e^{-iy^+})} \\
&= -\frac{e^{i(y^- + y^+)} + e^{-i(y^- + y^+)} - (e^{i(y^- - y^+)} + e^{-i(y^- - y^+)})}{e^{i(y^- + y^+)} + e^{-i(y^- + y^+)} + e^{i(y^- - y^+)} + e^{-i(y^- - y^+)}} \\
&= \frac{\cos(y^- - y^+) - \cos(y^- + y^+)}{\cos(y^- - y^+) + \cos(y^- + y^+)}
\end{aligned}$$

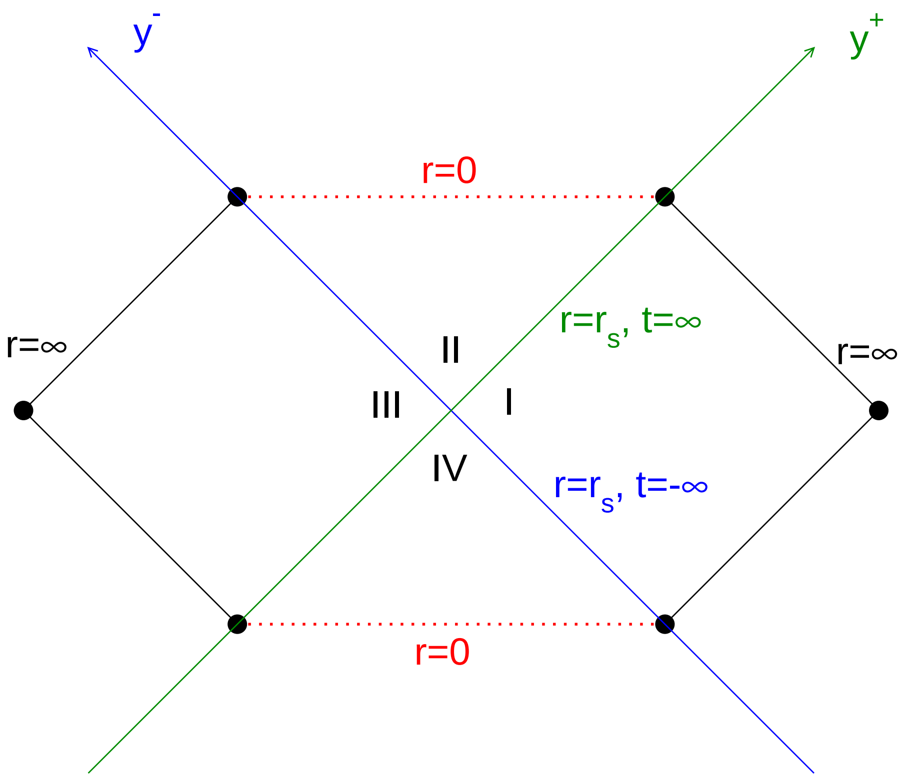
This condition is fulfilled if the second term in the numerator and denominator vanishes, so

$$\cos(y^- + y^+) = 0$$

which translates to

$$y^- + y^+ = \pm\frac{\pi}{2}$$

This is the upper and lower boundary of spacetime in the Penrose diagram, corresponding to the singularity.



2.3 The Kerr black hole

Although the Schwarzschild solution is interesting and exhibits many strikingly new aspects of general relativity, its applicability to black holes that are remnants of stellar collapse is quite limited. Stars are spinning and although they can lose angular momentum, when they contract to their Schwarzschild radius, the rotation velocity at the surface may reach a sizeable fraction of the speed of light. We therefore need a better description of these objects that are still axially symmetric, but rotate around their axis.

2.3.1 Boyer-Lindquist coordinates

In 1963 Kerr found a metric that describes a rotating eternal black hole. As in the case of the Schwarzschild solution, one can check that it is in fact a vacuum solution, so $R_{\mu\nu}=0$ on the entire spacetime that the solution is defined. We will, as usual, write down the metric first and work from there. The metric is given by

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s r}{\rho^2}\right)\mathrm{d}t^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 - \frac{\Sigma}{\rho^2}\sin^2\theta\,\mathrm{d}\varphi^2 + \frac{2a r_s r}{\rho^2}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi \tag{28}$$

where we used the shorthand

$$\rho^2 = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - r_s r + a^2 \qquad \Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$$

The most striking feature of this metric is the new non-diagonal term in the metric, which couples the angular coordinate φ with the asymptotic time coordinate t. The (dimensionful) parameter that controls this coupling is a, so let us first see what happens if we let $a\to 0$. In this limit, the $\mathrm{d}t\mathrm{d}\varphi$ term vanishes while $\rho = r$, $\Sigma = r^4$ and $\Delta = r(r-r_s)$. This evidently results in the Schwarzschild metric, so this first limit is correct. Another important property concerns the asymptotic observer. Since we now have two parameters of dimension distance in the metric, the asymptotic observer conditions now are $r \gg r_s$ as well as $r \gg a$. The second of these conditions again implies the vanishing of the $\mathrm{d}t\mathrm{d}\varphi$ term as well as $\rho\to r$, $\Sigma\to r^4$ and $\Delta\to r(r-r_s)$. Therefore, the asymptotic observer has the same metric as in the Schwarzschild case, i.e. flat Minkovski.

The next thing we would like to check is whether this metric really is stationary, i.e. whether there is a timelike killing vector that ensures time translation symmetry. Let us see if a shift in the coordinate t accomplishes this, i.e. whether the vector

$$t^\alpha = x^\alpha{}_{,t} \overset{*}{=} \delta^\alpha_t$$

is indeed a Killing vector. When we write the Killing equation in the form

$$\mathcal{L}_t g_{\mu\nu} = t^\alpha g_{\mu\nu,\alpha} + t^\alpha{}_{,\mu} g_{\alpha\nu} + t^\alpha{}_{,\nu} g_{\mu\alpha} = 0$$

we can immediately see that it is fulfilled. The first term vanishes because the metric is not dependent on t, while the second and third terms vanish because t^α is constant in our coordinates. In a very similar fashion we can check that the vector

$$\varphi^\alpha = x^\alpha{}_{,\varphi} \overset{*}{=} \delta^\alpha_\varphi$$

is also a Killing vector. This is the axial symmetry of the metric and the corresponding conserved quantity $u^\alpha\varphi_\alpha$ is thus the angular momentum of an affinely parameterised geodesic with tangent vector u^α. If we take the special case of a geodesic with vanishing angular momentum, we have

$$\begin{aligned} 0 &= u^\alpha\varphi_\alpha \\ &\overset{*}{=} u^\alpha g_{\alpha\beta}\delta^\beta_\varphi \\ &= u^\alpha g_{\alpha\varphi} \\ &= u^t g_{t\varphi} + u^\varphi g_{\varphi\varphi} \\ &= \dot{t}\frac{a r_s r}{\rho^2}\sin^2\theta - \dot{\varphi}\frac{\Sigma}{\rho^2}\sin^2\theta \end{aligned}$$

where the overdot represents the derivative with respect to the affine parameter of the geodesic. From this relation we obtain

$$\dot{t} a r_s r = \dot{\varphi} \Sigma$$

and thus

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{a r_s r}{\Sigma} \tag{29}$$

This is a rather remarkable result. It tells us that freely falling observers with no angular momentum are in fact rotating in a Kerr metric. This phenomenon is known as frame dragging or the Lense-Thirring effect. If we expand Σ in the previous equation, we can see that the angular velocity

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{a r_s r}{(r^2 + a^2)^2 - a^2(r^2 - r_s r + a^2)\sin^2\theta} \xrightarrow{r \to \infty} \frac{a r_s}{r^3}$$

vanishes for an asymptotic observer as r^{-3}.

2.3.2 Ergosurface and horizons

We now look at an observer at fixed coordinates r, θ and φ as we did in the Schwarzschild case. If we take two points at t and $t + \mathrm{d}t$, we can see that they are separated by

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s r}{\rho^2}\right)\mathrm{d}t^2 = \left(1 - \frac{r_s r}{r^2 + a^2\cos^2\theta}\right)\mathrm{d}t^2$$

As in the Schwarzschild case, we want to identify a point where the t coordinate turns from timelike into null and then spacelike. This happens when

$$r^2 + a^2\cos^2\theta = r_s r \tag{30}$$

so

$$r = \frac{r_s \pm \sqrt{r_s^2 - 4a^2\cos^2\theta}}{2}$$

Remarkably, this has real solutions for all θ only if $a \leqslant r_s/2$. If this condition is fulfilled however, we now have a second solution to this equation where the distance turns from timelike to null and again back to spacelike. For the moment we will assume $a \leqslant r_s/2$ and only look at the outer solution

$$r_e = \frac{r_s}{2} + \sqrt{\left(\frac{r_s}{2}\right)^2 - a^2\cos^2\theta} \tag{31}$$

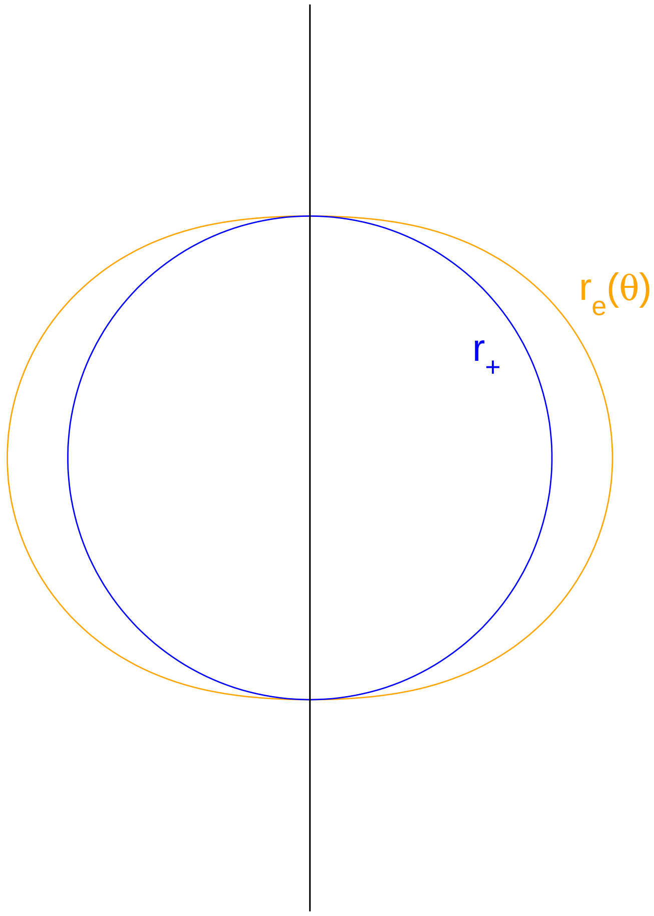
At this surface, the time dilation for an asymptotic observer is infinite. It is therefore referred to as the infinite redshift surface. We are tempted to interpret it as a horizon and in order to check whether it really is one, we look at two events that have common coordinates t, θ and φ and are separated in r by $\mathrm{d}r$. The distance between these points is

$$\mathrm{d}\tau^2 = -\frac{\rho^2}{\Delta}\mathrm{d}r^2 = -\frac{r^2 + a^2\cos^2\theta}{r^2 - r_s r + a^2}\mathrm{d}r^2$$

For large r this distance is spacelike. It diverges at

$$\Delta = r^2 - r_s r + a^2 = 0 \quad \Rightarrow \quad r_\pm = \frac{r_s}{2} \pm \sqrt{\left(\frac{r_s}{2}\right)^2 - a^2} \tag{32}$$

If we first look at the larger of these two solutions, r_+, we see, that it only agrees with (31) at the poles $\theta = \pm\pi/2$. How can we understand this discrepancy?



We get a clear hint about the discrepancy from the fact that it vanishes at the poles. At the poles we have no frame dragging effect and there r_e, which indicates the point from where on no observer can remain static, coincides with r_+, which indicates that time now runs in the r coordinate. Clearly we have a region (called ergosphere), where observers can not remain static, but do not necessarily need to proceed towards smaller radii yet. We suspect, that inside the ergosphere (which incidentally is not a sphere at all) the inability to remain static is due to frame dragging. It is so strong that observers necessarily have to alter their position in φ, but not necessarily in r. The outer bound of the ergosphere, the surface of infinite redshift, is also called the ergosurface.

2.3.3 Inside the ergosphere

We have seen that inside the ergosphere two events, which share common coordinates r, θ and φ and have t coordinates that differ by $\mathrm{d}t$, are separated in a spacelike manner. If it is true that this inability to remain static is due to the strong frame dragging, we should allow the events to also be displaced in the φ direction. We thus look at the separation of two events with common r and θ that are separated by $\mathrm{d}t$ and $\mathrm{d}\varphi = \omega \mathrm{d}t$. We would like to find the angular velocity ω for which their separation is maximally timelike. From the metric (28) we get

$$\mathrm{d}\tau^2 = \mathrm{d}t^2\left(1 - \frac{r_s r}{\rho^2} - \frac{\Sigma}{\rho^2}\sin^2\theta\omega^2 + \frac{2ar_s r}{\rho^2}\sin^2\theta\omega\right) \tag{33}$$

We maximise the bracketed expression with respect to ω, so

$$-2\frac{\Sigma}{\rho^2}\sin^2\theta\omega + \frac{2ar_s r}{\rho^2}\sin^2\theta = 0$$

which has the solution

$$\omega = \frac{ar_s r}{\Sigma}$$

that unsurprisingly coincides with the angular velocity (29) that we found from the frame dragging. When we plug ω back into (33) we obtain

$$
\begin{aligned}
\mathrm{d}\tau^2 &= \mathrm{d}t^2\left(1-\frac{r_s r}{\rho^2}-\frac{\Sigma}{\rho^2}\sin^2\theta\left(\frac{a\,r_s\,r}{\Sigma}\right)^2+\frac{2a\,r_s r}{\rho^2}\sin^2\theta\,\frac{a\,r_s\,r}{\Sigma}\right)\\
&= \mathrm{d}t^2\left(1-\frac{r_s r}{\rho^2}+\frac{1}{\rho^2}\sin^2\theta\,\frac{a^2\,r_s^2\,r^2}{\Sigma}\right)\\
&= \mathrm{d}t^2\frac{\Sigma(\rho^2-r_s r)+a^2\,r_s^2\,r^2\sin^2\theta}{\Sigma\rho^2}\\
&= \mathrm{d}t^2\frac{\Sigma(r^2+a^2\cos^2\theta+\Delta-r^2-a^2)+a^2\,(r^2+a^2-\Delta)^2\sin^2\theta}{\Sigma\rho^2}\\
&= \mathrm{d}t^2\frac{\Sigma(\Delta-a^2\sin^2\theta)+a^2\,(r^2+a^2-\Delta)^2\sin^2\theta}{\Sigma\rho^2}\\
&= \mathrm{d}t^2\frac{\Sigma\Delta+a^2\sin^2\theta((r^2+a^2-\Delta)^2-\Sigma)}{\Sigma\rho^2}\\
&= \mathrm{d}t^2\frac{\Sigma\Delta+a^2\sin^2\theta((r^2+a^2)^2-2\Delta(r^2+a^2)+\Delta^2-(r^2+a^2)^2+a^2\Delta\sin^2\theta)}{\Sigma\rho^2}\\
&= \mathrm{d}t^2\frac{\Sigma\Delta+a^2\sin^2\theta(-2\Delta(r^2+a^2)+\Delta^2+a^2\Delta\sin^2\theta)}{\Sigma\rho^2}\\
&= \mathrm{d}t^2\frac{\Sigma+a^2\sin^2\theta(-2(r^2+a^2)+\Delta+a^2\sin^2\theta)}{\Sigma\rho^2}\Delta\\
&= \mathrm{d}t^2\frac{\Sigma+a^2\sin^2\theta\Delta+a^2\sin^2\theta(-2(r^2+a^2)+a^2\sin^2\theta)}{\Sigma\rho^2}\Delta\\
&= \mathrm{d}t^2\frac{(r^2+a^2)^2+a^2\sin^2\theta(-2(r^2+a^2)+a^2\sin^2\theta)}{\Sigma\rho^2}\Delta\\
&= \mathrm{d}t^2\frac{r^2(r^2+a^2)+a^2(r^2+a^2)+a^2\sin^2\theta(-2(r^2+a^2)+a^2\sin^2\theta)}{\Sigma\rho^2}\Delta\\
&= \mathrm{d}t^2\frac{r^2(r^2+a^2)+a^2(r^2+a^2)(1-2\sin^2\theta)+(a^2\sin^2\theta)^2)}{\Sigma\rho^2}\Delta\\
&= \mathrm{d}t^2\frac{r^4+r^2a^2+a^2r^2(1-2\sin^2\theta)+a^4(1-2\sin^2\theta+\sin^4\theta))}{\Sigma\rho^2}\Delta\\
&= \mathrm{d}t^2\frac{r^4+a^2r^2(2-2\sin^2\theta)+a^4(1-\sin^2\theta)^2}{\Sigma\rho^2}\Delta\\
&= \mathrm{d}t^2\frac{r^4+2r^2(a^2\cos^2\theta)+(a^2\cos^2\theta)^2}{\Sigma\rho^2}\Delta\\
&= \mathrm{d}t^2\frac{(r^2+a^2\cos^2\theta)^2}{\Sigma\rho^2}\Delta\\
&= \mathrm{d}t^2\frac{\rho^2}{\Sigma}\Delta
\end{aligned}
$$

Because $\Sigma>0$, we arrive at a lightlike separation $\mathrm{d}\tau=0$ for the two points when $\Delta=0$. From (32) we already know that the solutions to this equation

$$r_\pm=\frac{r_s}{2}\pm\sqrt{\left(\frac{r_s}{2}\right)^2-a^2}$$

are the radii where r becomes a lightlike coordinate, too. So even if an observer optimally corotates with an angular velocity ω, progressing forward in t will not be timelike anymore if the outer of the two radii r_+ is crossed. We can thus conclude that r_+ correspond to a horizon where spacelike and timelike coordinates are exchanged, very similar to the Schwarzschild case. We will shortly discuss what happens at the inner of the two radii r_-, but before that let us investigate the ergosphere, i.e. the region where $r_+<r<r_e$ a bit more. For this purpose we return to (33) but without assuming that we corotate with the optimal angular velocity. Generally, observers that rotate with an arbitrary angular velocity but stay at fixed r and θ are called stationary. For such a stationary observer, we can find for which combination of radius r and angular velocity Ω going

forward in t will become lightlike. Reading off directly from (33) we obtain

$$1-\frac{r_s r}{\rho^2}-\frac{\Sigma}{\rho^2}\sin^2\theta\,\Omega^2+\frac{2a\,r_s r}{\rho^2}\sin^2\theta\,\Omega=0$$

which has the solutions

$$\begin{aligned}
\Omega_{\mp} &= \frac{-\frac{2a\,r_s r}{\rho^2}\sin^2\theta\pm\sqrt{\left(\frac{2a\,r_s r}{\rho^2}\sin^2\theta\right)^2+4\frac{\Sigma}{\rho^2}\sin^2\theta\left(1-\frac{r_s r}{\rho^2}\right)}}{2\left(-\frac{\Sigma}{\rho^2}\sin^2\theta\right)}\\
&= \frac{a r_s r\mp\sqrt{(a r_s r)^2+\Sigma\sin^{-2}\theta(\rho^2-r_s r)}}{\Sigma}\\
&= \frac{a r_s r}{\Sigma}\mp\frac{\sqrt{(a r_s r\sin\theta)^2+\Sigma\,(\rho^2-r_s r)}}{\Sigma\sin\theta}\\
&= \omega\mp\frac{\sqrt{a^2(r^2+a^2-\Delta)^2\sin^2\theta+((r^2+a^2)^2-a^2\Delta\sin^2\theta)(r^2+a^2\cos^2\theta-r^2-a^2+\Delta)}}{\Sigma\sin\theta}\\
&= \omega\mp\frac{\sqrt{((r^2+a^2)^2-2\Delta(r^2+a^2)+\Delta^2)a^2\sin^2\theta+((r^2+a^2)^2-a^2\Delta\sin^2\theta)(-a^2\sin^2\theta+\Delta)}}{\Sigma\sin\theta}\\
&= \omega\mp\frac{\sqrt{(-2\Delta(r^2+a^2)+\Delta^2)a^2\sin^2\theta+(-a^2\Delta\sin^2\theta)(-a^2\sin^2\theta+\Delta)+\Delta(r^2+a^2)^2}}{\Sigma\sin\theta}\\
&= \omega\mp\frac{\sqrt{\Delta}\sqrt{(a^2\sin^2\theta)^2-2(r^2+a^2)a^2\sin^2\theta+(r^2+a^2)^2}}{\Sigma\sin\theta}\\
&= \omega\mp\frac{\sqrt{\Delta}\sqrt{(r^2+a^2-a^2\sin^2\theta)^2}}{\Sigma\sin\theta}\\
&= \omega\mp\frac{\sqrt{\Delta}(r^2+a^2\cos^2\theta)}{\Sigma\sin\theta}\\
&= \omega\mp\frac{\sqrt{\Delta}\rho^2}{\Sigma\sin\theta}
\end{aligned}$$

Thus for an angular velocity Ω in between Ω_- and Ω_+, $\mathrm{d}\tau^2$ is timelike. In the trivial, flat space case $r_s=a=0$ we have $\Omega_\pm=\pm\frac{1}{r\sin\theta}$. This limit is easy to understand: If one rotates with an angular velocity Ω exceeding Ω_+, the linear velocity in φ direction $v=\Omega r\sin\theta$ would exceed the speed of light. In the Schwarzschild case ($a=0$), this relation gets modified to

$$\Omega_\pm=\pm\sqrt{1-\frac{r_s}{r}}\frac{1}{r\sin\theta}$$

but the allowed angular velocities are still symmetric around $\Omega=0$. In case of the Kerr metric however, the allowed angular velocities center around ω. Therefore there is a point where the lower limit of the allowed angular velocity Ω_- becomes positive. This happens at

$$\Omega_-=\frac{a r_s r}{\Sigma}-\frac{\sqrt{\Delta}\rho^2}{\Sigma\sin\theta}=0$$

or

$$a^2 r_s^2 r^2\sin^2\theta=\Delta\rho^4$$

We can recast this into

$$\begin{aligned}
0 &= a^2 r_s^2 r^2\sin^2\theta-\Delta\rho^4\\
&= (r^2+a^2-\Delta)^2a^2\sin^2\theta-\Delta(r^2+a^2\cos^2\theta)^2\\
&= (r^2+a^2-\Delta)^2a^2\sin^2\theta-\Delta(r^2+a^2-a^2\sin^2\theta)^2\\
&= (r^2+a^2)^2a^2\sin^2\theta+\Delta^2a^2\sin^2\theta-\Delta(r^2+a^2)^2-\Delta(a^2\sin^2\theta)^2\\
&= (r^2+a^2)^2(a^2\sin^2\theta-\Delta)+\Delta a^2\sin^2\theta(\Delta-a^2\sin^2\theta)\\
&= (a^2\sin^2\theta-\Delta)((r^2+a^2)^2-a^2\Delta\sin^2\theta)\\
&= (a^2\sin^2\theta-r^2+r_s r-a^2)\Sigma\\
&= -(r^2+a^2\cos^2\theta-r_s r)\Sigma
\end{aligned}$$

Since $\Sigma > 0$ it follows that

$$r^2 + a^2 \cos^2\theta = r_s r$$

which we recognize is just the condition (30) for the outer boundary of the ergosphere. Thus we see explicitly that the ergosphere is the region where the metric rotates so fast that staying at the same angular coordinate would require a local velocity exceeding the speed of light.

2.3.4 The Penrose process

Except for a factor m, the conserved quantity $e = u^\mu t_\mu = u^\mu g_{\mu t}$ is the energy of a test mass. In flat space it is $e = \dot{t}$ and therefore always positive for a forward directed, timelike trajectory. In the Schwarzschild case, we have $e = \dot{t}\left(1 - \frac{r_s}{r}\right)$ and therefore it is always positive for a forward directed, timelike trajectory outside the Schwarzschild radius. For the Kerr metric we have

$$\begin{aligned} e &= \left(1 - \frac{r_s r}{\rho^2}\right)\dot{t} + \frac{a r_s r}{\rho^2}\sin^2\theta\,\dot{\varphi} \\ &= \left(1 - \frac{r_s r}{\rho^2}(1 - a\Omega\sin^2\theta)\right)\dot{t} \end{aligned}$$

with the angular velocity of the test mass Ω. We already know that the ergosurface is the outermost point at which $\Omega = 0$ is allowed. We also know from (30) that at the ergosurface $r_s r = \rho^2$. All this tells us that at the ergosurface we can have $e = 0$. Inside the ergosphere we can have negative energy states, just as we could have had negative energy particles inside the horizon of a Schwarzschild black hole. There is however one difference: Nothing could escape from the region inside the Schwarzschild black hole, so the potential existence of negative energy particles was irrelevant for outside observers. In the case of the Kerr metric that is different: The ergosurface is not a horizon, objects can enter it and escape again. So one could imagine that an object enters the ergosphere, interacts with an object there, giving that object more negative energy, and then emerges from the ergosphere with more energy than it entered. The other object, which has $e < 0$ is thus trapped and can never exit the ergosphere. In this manner, one can in principle extract energy from a Kerr black hole in a fully classical manner. This phenomenon is called the Penrose process.

2.3.5 Circular orbits in the equatorial plane

Let us examine some circular orbits $\dot{r} = 0$, $\theta = \pi/2$. The geodesic equations are

$$\begin{aligned} \ddot{t} &= 0 \\ \Gamma^r_{tt}\dot{t}^2 + 2\Gamma^r_{t\varphi}\dot{t}\dot{\varphi} + \Gamma^r_{\varphi\varphi}\dot{\varphi}^2 &= 0 \\ \Gamma^\theta_{tt}\dot{t}^2 + 2\Gamma^\theta_{t\varphi}\dot{t}\dot{\varphi} + \Gamma^\theta_{\varphi\varphi}\dot{\varphi}^2 &= 0 \\ \ddot{\varphi} &= 0 \end{aligned}$$

and the relevant Christoffel symbols (for $\theta = \pi/2$) read

$$\begin{aligned} \Gamma^r_{tt} &= \frac{(a^2 + r(r - r_s))r_s}{2r^4} \\ \Gamma^r_{t\varphi} &= -\frac{a(a^2 + r(r - r_s))r_s}{2r^4} \\ \Gamma^r_{\varphi\varphi} &= \frac{(a^2 + r(r - r_s))(-2r^4 + a^2 r r_s)}{2r^5} \\ \Gamma^\theta_{tt} &= 0 \\ \Gamma^\theta_{t\varphi} &= 0 \\ \Gamma^\theta_{\varphi\varphi} &= 0 \end{aligned}$$

The only nontrivial equation is thus

$$\Gamma^r_{tt}\left(\dot{t}^2 - 2a\dot{t}\dot{\varphi} + \frac{-2r^4 + a^2 r r_s}{r r_s}\dot{\varphi}^2\right) = 0$$

which has the solution

$$\begin{aligned}\dot{t} &= a\dot{\varphi} \pm \sqrt{a^2\dot{\varphi}^2 - \frac{-2r^4 + a^2 r r_s}{r r_s}\dot{\varphi}^2} \\ &= \dot{\varphi}\left(a \pm \sqrt{\frac{2r^3}{r_s}}\right) \\ &= \dot{\varphi}\left(a \pm r\sqrt{\frac{2r}{r_s}}\right)\end{aligned}$$

or

$$\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{a \pm r\sqrt{\frac{2r}{r_s}}}$$

At $\theta = \pi/2$, the ergosphere stretches from $r_+ = \frac{r_s}{2} + \sqrt{\left(\frac{r_s}{2}\right)^2 - a^2}$ to $r_e = r_s$. At its outer edge the smaller of the two angular frequencies is thus

$$\Omega_e = \frac{1}{a + r_e\sqrt{\frac{2r_e}{r_s}}} = \frac{1}{a + r_s\sqrt{2}}$$

Let us check if this is smaller than the maximally allowed frequency Ω_+:

$$\begin{aligned}\frac{\Omega_+}{\Omega_e} &= (a + r_s\sqrt{2})\frac{a r_s^2 + \sqrt{\Delta}\rho^2}{\Sigma} \\ &= (a + r_s\sqrt{2})\frac{a r_s^2 + a r_s^2}{(r_s^2 + a^2)^2 - a^4} \\ &= 2(a + r_s\sqrt{2})\frac{a r_s^2}{r_s^4 + 2a^2 r_s^2} \\ &= 2a\frac{a + r_s\sqrt{2}}{r_s^2 + 2a^2}\end{aligned}$$

This ratio should be larger than one, which implies

$$r_s^2 + 2a^2 < 2a^2 + 2\sqrt{2}a r_s$$

or

$$r_s < 2\sqrt{2}a \qquad \Rightarrow \qquad a > \frac{\sqrt{2}}{4}r_s$$

If this condition is fulfilled, the Kerr metric allows for circular orbits inside the ergosphere.

2.3.6 Geodesics in the equatorial plane

Let us now look at arbitrary geodesics in the equatorial plane. In this case we have $\theta = \pi/2$ (so $\mathrm{d}\theta = 0$) and thus the metric reduces to

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \frac{r^2}{\Delta}\mathrm{d}r^2 - \frac{\Sigma}{r^2}\mathrm{d}\varphi^2 + \frac{2a r_s}{r}\mathrm{d}t\,\mathrm{d}\varphi$$

with

$$\Delta = r^2 - r_s r + a^2 \qquad \Sigma = (r^2 + a^2)^2 - a^2\Delta$$

Let us write the relevant metric elements:

$$\begin{aligned}
g_{tt} &= 1-\frac{r_s}{r}\\
g_{rr} &= -\frac{1}{1-\frac{r_s}{r}+\frac{a^2}{r^2}}\\
g_{\varphi\varphi} &= -r^2-a^2\Big(1+\frac{r_s}{r}\Big)\\
g_{t\varphi} &= \frac{a\,r_s}{r}
\end{aligned}$$

We might again proceed by writing down the geodesic equations explicilty, which is however rather complicated. Let us instead use our knowledge of Killing vecrors t^α and φ^α and look at the corresponding conserved quantities. The first one, with is proportional to energy and we therefore call e, can be expressed as

$$\begin{aligned}
e &= u^\mu t_\mu\\
&\stackrel{*}{=} u^\mu g_{\mu\nu}\delta^\nu_t\\
&= u^t g_{tt}+u^\varphi g_{\varphi t}\\
&= \dot{t}g_{tt}+\dot{\varphi}g_{\varphi t}
\end{aligned}$$

The second, which is proportional to angular momentum, reads

$$\begin{aligned}
\ell &= -u^\mu\varphi_\mu\\
&\stackrel{*}{=} -u^\mu g_{\mu\nu}\delta^\nu_\varphi\\
&= -u^t g_{t\varphi}-u^\varphi g_{\varphi\varphi}\\
&= -\dot{t}g_{t\varphi}-\dot{\varphi}g_{\varphi\varphi}
\end{aligned}$$

Thus we can express the t and φ components of the tangent vector of the geodesic as

$$\begin{aligned}
\dot{t} &= \frac{g_{\varphi\varphi}e+g_{\varphi t}\ell}{g_{tt}g_{\varphi\varphi}-g_{\varphi t}^2}\\
&= \frac{\big(-r^2-a^2\big(1+\frac{r_s}{r}\big)\big)e+\frac{a\,r_s}{r}\ell}{\big(1-\frac{r_s}{r}\big)\big(-r^2-a^2\big(1+\frac{r_s}{r}\big)\big)-\big(\frac{a\,r_s}{r}\big)^2}\\
&= \frac{(r^2+a^2)e-\frac{a\,r_s}{r}(\ell-ae)}{a^2\big(1-\big(\frac{r_s}{r}\big)^2\big)+\big(1-\frac{r_s}{r}\big)r^2+\big(\frac{a\,r_s}{r}\big)^2}\\
&= \frac{(r^2+a^2)e-\frac{a\,r_s}{r}(\ell-ae)}{a^2-r_sr+r^2}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\varphi} &= \frac{g_{t\varphi}e+g_{tt}\ell}{g_{\varphi t}^2-g_{tt}g_{\varphi\varphi}}\\
&= \frac{\frac{a\,r_s}{r}e+\big(1-\frac{r_s}{r}\big)\ell}{\Delta}
\end{aligned}$$

where we have used the identity

$$g_{\varphi t}^2-g_{tt}g_{\varphi\varphi}=a^2-r_sr+r^2=\Delta$$

The equation for the only remaining nontrivial component $\dot{r}$ may now simply be obtained by the normalization condition of the tangential vector

$$\begin{aligned}
1 &= u^\mu u_\mu\\
&= u^\mu g_{\mu\nu}u^\nu\\
&= g_{tt}\dot{t}^2+g_{rr}\dot{r}^2+g_{\varphi\varphi}\dot{\varphi}^2+2g_{t\varphi}\dot{t}\dot{\varphi}
\end{aligned}$$

This implies

$$
\begin{aligned}
\dot r^2 &= \frac{1-g_{tt}\dot t^2 - g_{\varphi\varphi}\dot\varphi^2 - 2g_{t\varphi}\dot t\dot\varphi}{g_{rr}}\\
&= \frac{1-g_{tt}\Big(\frac{g_{\varphi\varphi}e+g_{\varphi t}\ell}{g_{tt}g_{\varphi\varphi}-g_{\varphi t}^2}\Big)^2 - g_{\varphi\varphi}\Big(\frac{g_{t\varphi}e+g_{tt}\ell}{g_{\varphi t}^2-g_{tt}g_{\varphi\varphi}}\Big)^2 - 2g_{t\varphi}\frac{g_{\varphi\varphi}e+g_{\varphi t}\ell}{g_{tt}g_{\varphi\varphi}-g_{\varphi t}^2}\frac{g_{t\varphi}e+g_{tt}\ell}{g_{\varphi t}^2-g_{tt}g_{\varphi\varphi}}}{-\frac{r^2}{\Delta}}\\
&= -\frac{\Delta}{r^2} + \frac{-g_{tt}(g_{\varphi\varphi}e+g_{\varphi t}\ell)^2 - g_{\varphi\varphi}(g_{t\varphi}e+g_{tt}\ell)^2 + 2g_{t\varphi}(g_{\varphi\varphi}e+g_{\varphi t}\ell)(g_{t\varphi}e+g_{tt}\ell)}{-\frac{r^2}{\Delta}\Delta^2}\\
&= -\frac{\Delta}{r^2} + \frac{-g_{tt}g_{\varphi\varphi}^2e^2 - 2g_{tt}g_{\varphi\varphi}g_{\varphi t}\ell e - g_{tt}g_{\varphi t}^2\ell^2 - g_{\varphi\varphi}g_{t\varphi}^2e^2 - 2g_{\varphi\varphi}g_{t\varphi}g_{tt}\ell e - g_{\varphi\varphi}g_{tt}^2\ell^2}{-r^2\Delta}\\
&\quad + \frac{2g_{\varphi\varphi}g_{t\varphi}^2e^2 + 2g_{\varphi t}^2g_{t\varphi}e\ell + 2g_{t\varphi}g_{\varphi\varphi}g_{tt}\ell e + 2g_{\varphi t}^2g_{tt}\ell^2}{-r^2\Delta}\\
&= -\frac{\Delta}{r^2} + \frac{g_{tt}g_{\varphi\varphi}^2e^2 + 2(g_{tt}g_{\varphi\varphi}-g_{\varphi t}^2)g_{\varphi t}\ell e - g_{tt}g_{\varphi t}^2\ell^2 - g_{\varphi\varphi}g_{t\varphi}^2e^2 + g_{\varphi\varphi}g_{tt}^2\ell^2}{r^2\Delta}\\
&= -\frac{\Delta}{r^2} + \frac{g_{\varphi\varphi}(g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2)e^2 + g_{tt}(g_{\varphi\varphi}g_{tt}-g_{\varphi t}^2)\ell^2 - 2\Delta g_{\varphi t}\ell e}{r^2\Delta}\\
&= -\frac{\Delta}{r^2} + \frac{-\Delta g_{\varphi\varphi}e^2 - \Delta g_{tt}\ell^2 - 2\Delta g_{\varphi t}\ell e}{r^2\Delta}\\
&= \frac{-\Delta - g_{\varphi\varphi}e^2 - g_{tt}\ell^2 - 2g_{\varphi t}\ell e}{r^2}\\
&= \frac{-(a^2-r_sr+r^2) + \big(r^2+a^2\big(1+\frac{r_s}{r}\big)\big)e^2 - 2\frac{ar_s}{r}e\ell - \big(1-\frac{r_s}{r}\big)\ell^2}{r^2}
\end{aligned}
$$

so, in summary, an equatorial geodesic in terms of the constants of motion is described by

$$
\begin{aligned}
\dot t &= \frac{(r^2+a^2)e - \frac{ar_s}{r}(\ell - ae)}{a^2-r_sr+r^2}\\
\dot r &= \pm\frac{1}{r}\sqrt{-(a^2-r_sr+r^2) + \Big(r^2+a^2\Big(1+\frac{r_s}{r}\Big)\Big)e^2 - 2\frac{ar_s}{r}e\ell - \Big(1-\frac{r_s}{r}\Big)\ell^2}\\
\dot\varphi &= \frac{\frac{ar_s}{r}e + \big(1-\frac{r_s}{r}\big)\ell}{a^2-r_sr+r^2}
\end{aligned}
$$

2.3.7 Radial freefall

Let us now investigate the simple case of an observer which is radially free falling into the Kerr black hole along the equator. Let us simplify the situation even further by assuming that the free fall trajectory started with an asymptotic observer at rest in the infinite past. At that time we had $\dot t = 1$ as well as $\dot r = \dot\varphi = 0$. Consequently the conserved quantities are $e = 1$ and $\ell = 0$. This in turn simplifies the geodesic equations to

$$
\begin{aligned}
\dot t &= \frac{r^2+a^2+\frac{a^2r_s}{r}}{r^2+a^2-r_sr}\\
\dot r &= -\sqrt{\frac{r_s}{r^3}(r^2+a^2)}\\
\dot\varphi &= \frac{ar_s}{r(a^2-r_sr+r^2)}
\end{aligned}
\tag{34}
$$

As we have already seen, a radially ingalling observer at vanishing angular momentum will still not be at a fixed φ coordinate. We are however interested primarily in the radial part of the motion. In the observers eigentime we have

$$
\mathrm{d}\tau = -\frac{1}{\sqrt{r_s}}r^{\frac{3}{2}}\frac{\mathrm{d}r}{\sqrt{r^2+a^2}}
\tag{35}
$$

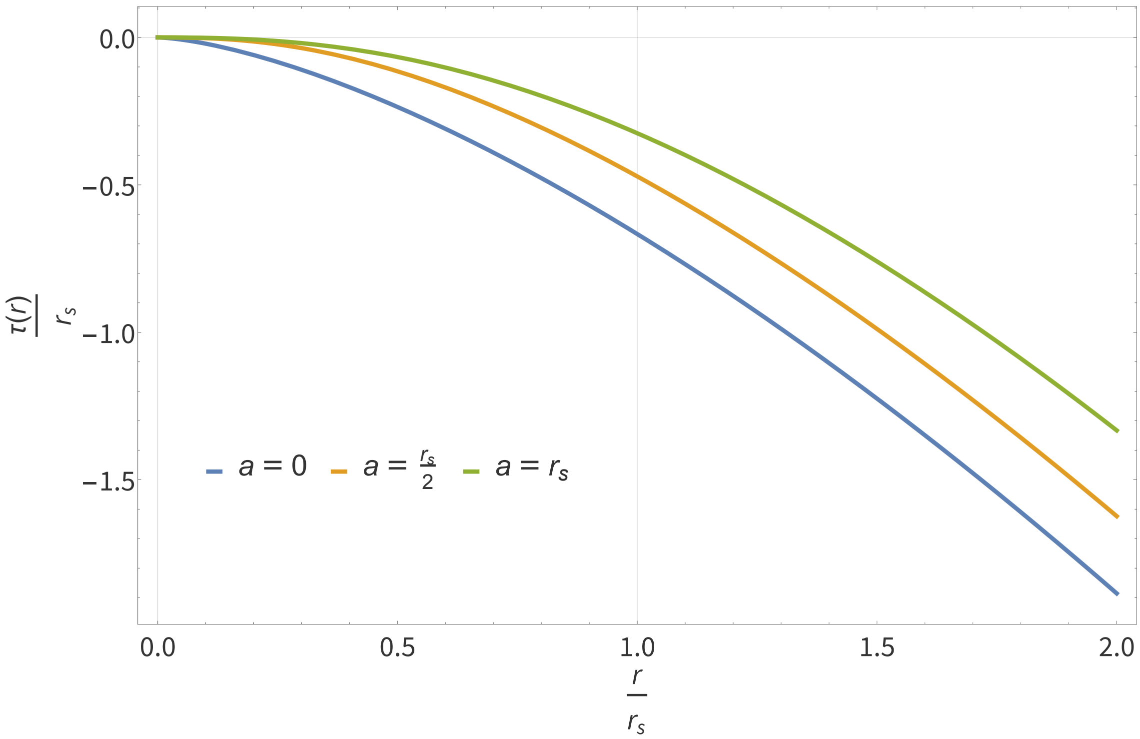
The integral on the right hand side results in a hypergeometric function

$$\begin{aligned}\tau_1-\tau_0 &= -\frac{1}{\sqrt{r_s}}\int_{r_0}^{r_1} r^{\frac{3}{2}}\frac{\mathrm{d}r}{\sqrt{r^2+a^2}}\\ &= -\frac{2\sqrt{r}}{3\sqrt{r_s}}\left(\sqrt{a^2+r^2}-a\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4},-\frac{r^2}{a^2}\right)\right)\Bigg|_{r=r_0}^{r=r_1}\end{aligned}$$

which stays finite as $r\to 0$. If we choose the integration constant such that $\tau=0$ at $r=0$, we have the relation

$$\tau=-\frac{2\sqrt{r}}{3\sqrt{r_s}}\left(\sqrt{a^2+r^2}-a\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4},-\frac{r^2}{a^2}\right)\right) \tag{36}$$

which we can plot for different values of $\frac{a}{r_s}$:



The case $a=0$ corresponds to the Schwarzschild case. As we can see, with larger values of a the radially infalling observer reaches the central singularity at $r=0$ even faster as measured in the observer's eigentime.

Let us now look at the same freefall in the asymptotic time coordinate t. Using the relation between eigentime and asymptotic observer time from (34) we obtain

$$\frac{\mathrm{d}t}{\mathrm{d}\tau}=\frac{r^2+a^2+\frac{a^2 r_s}{r}}{r^2+a^2-r_s r}=1+\frac{r_s r+\frac{a^2 r_s}{r}}{r^2+a^2-r_s r}$$

Introducing dimensionless variables

$$\hat{a}=\frac{a}{r_s}\qquad \hat{r}=\frac{r}{r_s}$$

we can rewrite this as

$$\mathrm{d}t=1+\frac{\hat{r}+\frac{\hat{a}^2}{\hat{r}}}{\hat{r}^2+\hat{a}^2-\hat{r}}\mathrm{d}\tau$$

Together with (35), which we can write as

$$\mathrm{d}\tau = -r_s \sqrt{\frac{\hat{r}^3}{\hat{r}^2 + \hat{a}^2}} \mathrm{d}\hat{r}$$

we thus obtain

$$\mathrm{d}t = -r_s \left(1 + \frac{1}{\hat{r}} \frac{\hat{r}^2 + \hat{a}^2}{\hat{r}^2 + \hat{a}^2 - \hat{r}}\right) \sqrt{\frac{\hat{r}^3}{\hat{r}^2 + \hat{a}^2}} \mathrm{d}\hat{r}$$

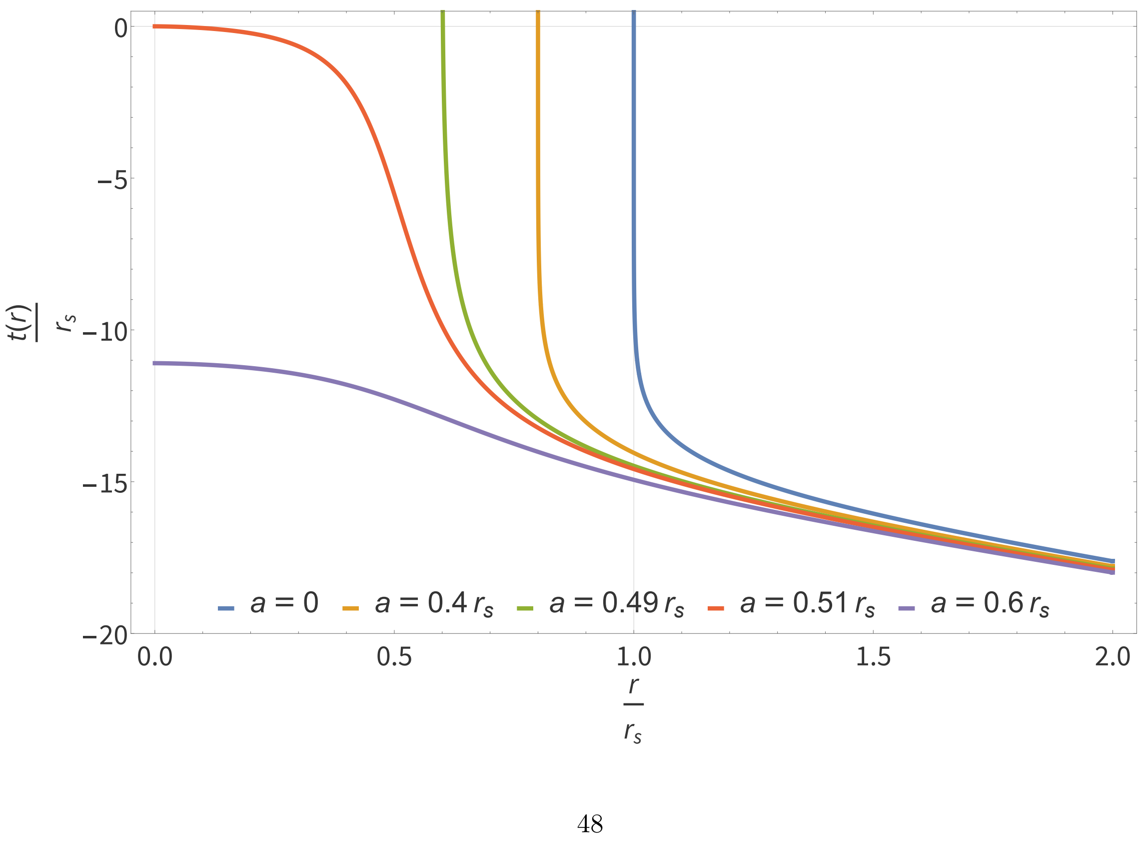
This expression diverges when

$$\hat{r}^2 + \hat{a}^2 - \hat{r} = \frac{\Delta}{r_s^2} = 0$$

According to (32), this happens at the horizon positions $r_\pm$. Since the freefall starts from outside, the first horizon encountered will be r_+. Thus, for the asymptotic observer, the infalling object will freeze at r_+ as expected. There is one catch however: As we know from (32), there will be no real solution to the equation $\Delta = 0$ for $a > r_s/2$. Let us write

$$\begin{aligned} \mathrm{d}t &= -r_s \left(1 + \frac{1}{\hat{r}} \frac{\hat{r}^2 + \hat{a}^2}{\hat{r}^2 + \hat{a}^2 - \hat{r}}\right) \sqrt{\frac{\hat{r}^3}{\hat{r}^2 + \hat{a}^2}} \mathrm{d}\hat{r} \\ &= -r_s \left(\sqrt{\frac{\hat{r}^3}{\hat{r}^2 + \hat{a}^2}} + \frac{1}{\hat{r}} \frac{\hat{r}^2 + \hat{a}^2}{\hat{r}^2 + \hat{a}^2 - \hat{r}} \sqrt{\frac{\hat{r}^3}{\hat{r}^2 + \hat{a}^2}}\right) \mathrm{d}\hat{r} \\ &= -r_s \left(\sqrt{\frac{\hat{r}^3}{\hat{r}^2 + \hat{a}^2}} + \sqrt{\hat{r}} \frac{\sqrt{\hat{r}^2 + \hat{a}^2}}{\hat{r}^2 + \hat{a}^2 - \hat{r}}\right) \mathrm{d}\hat{r} \end{aligned}$$

which can actually be integrated analytically in terms of elliptic and hypergeometric functions. It can however be easily integrated numerically, too. If we do this for various values of $\hat{a}$ and normalize the time such that at large radii they agree, we get the following functions $t(r)$:



We can see that for $\hat{a} > 1/2$, which translates into $a > r_s/2$ the asymptotic observer time for an infalling object to reach the singularity is finite. The singularity in this case is not shielded by a horizon anymore - a case known as naked singularity. There is a conjecture that such situations can not occur in nature - the cosmic censorship conjecture.

2.3.8 Principal null geodesics

Looking at the metric (28) again, we see that it has coordinate singularities at $\Delta = 0$ and $\rho = 0$. The singularity $\Delta = 0$ we have already identified as a horizon which we can hope to get rid of by a suitable coordinate transformation. The other coordinate singularity, at $\rho = 0$ has as a special case for $a = 0$ the Schwarzschild singularity at $r = 0$. We therefore expect this to be the true singularity of the Kerr metric - but it is rather strange. We have

$$r^2 + a^2 \cos^2\theta = 0$$

which can only be fulfilled if $r = 0$ **and** $\theta = \pi/2$. So in Boyer-Lindquist coordinates the singularity occurs when we approach one point along exactly the equatorial plane. This suggests that the true geometry of the singularity is not really a point and we need to change coordinates to examine this better. Let us start out by defining a vector field

$$l^\mu \overset{*}{=} \frac{r^2 + a^2}{\Delta}\delta^\mu_t - \delta^\mu_r + \frac{a}{\Delta}\delta^\mu_\varphi$$

Computing its norm

$$\begin{aligned}
l^\mu l_\mu &= l^\mu g_{\mu\nu} l^\nu \\
&= \left(\frac{r^2 + a^2}{\Delta}\right)^2 g_{tt} + g_{rr} + \frac{a^2}{\Delta^2} g_{\varphi\varphi} + 2\frac{r^2 + a^2}{\Delta}\frac{a}{\Delta} g_{t\varphi} \\
&= \frac{(r^2 + a^2)^2}{\Delta^2}\left(1 - \frac{r_s r}{\rho^2}\right) - \frac{\rho^2}{\Delta} - \frac{\Sigma}{\rho^2}\sin^2\theta \frac{a^2}{\Delta^2} + 2\frac{r^2 + a^2}{\Delta}\frac{a}{\Delta}\frac{a r_s r}{\rho^2}\sin^2\theta \\
&= \frac{1}{\Delta^2\rho^2}((r^2 + a^2)^2\rho^2 - \Delta\rho^4 - (r^2 + a^2)^2 r_s r - \Sigma\sin^2\theta a^2 + 2(r^2 + a^2)a^2 r_s r\sin^2\theta) \\
&= \frac{1}{\Delta^2\rho^2}((r^2 + a^2)^2(r^2 + a^2 - a^2\sin^2\theta) - \Delta(r^2 + a^2 - a^2\sin^2\theta)^2 \\
&\quad + (r^2 + a^2)r_s r(-r^2 - a^2 + 2a^2\sin^2\theta) - (r^2 + a^2)^2\sin^2\theta a^2 + \Delta(a^2\sin^2\theta)^2) \\
&= \frac{1}{\Delta^2\rho^2}((r^2 + a^2)^3 - (r^2 + a^2)^2 a^2\sin^2\theta - \Delta(r^2 + a^2)^2 - \Delta(a^2\sin^2\theta)^2 + 2\Delta(r^2 + a^2)a^2\sin^2\theta \\
&\quad + (r^2 + a^2)(r^2 + a^2 - \Delta)(-r^2 - a^2 + 2a^2\sin^2\theta) - (r^2 + a^2)^2\sin^2\theta a^2 + \Delta(a^2\sin^2\theta)^2) \\
&= \frac{1}{\Delta^2\rho^2}((r^2 + a^2)^3 - (r^2 + a^2)^2 a^2\sin^2\theta - \Delta(r^2 + a^2)^2 - \Delta(a^2\sin^2\theta)^2 + 2\Delta(r^2 + a^2)a^2\sin^2\theta \\
&\quad - (r^2 + a^2)^2(r^2 + a^2 - 2a^2\sin^2\theta) + (r^2 + a^2)\Delta(r^2 + a^2 - 2a^2\sin^2\theta)) \\
&\quad - (r^2 + a^2)^2\sin^2\theta a^2 + \Delta(a^2\sin^2\theta)^2) \\
&= 0
\end{aligned}$$

we see that it is a null vector. Next, we define

$$\mathrm{d}r^* = \frac{r^2 + a^2}{\Delta}\mathrm{d}r \qquad \mathrm{d}r^\# = \frac{a}{\Delta}\mathrm{d}r$$

which we can integrate as (see exercise sheet)

$$\begin{aligned} r^* &= r + \frac{r_s r_+}{\sqrt{r_s^2 - 4a^2}} \ln\left|\frac{r - r_+}{r_+}\right| - \frac{r_s r_-}{\sqrt{r_s^2 - 4a^2}} \ln\left|\frac{r - r_-}{r_-}\right| \\ r^\# &= \frac{a}{\sqrt{r_s^2 - 4a^2}} \ln\left|\frac{r - r_+}{r - r_-}\right| \end{aligned}$$

Let us now define new coordinates

$$v = t + r^* \qquad \psi = \varphi + r^\#$$

to replace t and φ. We see that

$$\mathrm{d}v = \mathrm{d}t + \mathrm{d}r^* = \mathrm{d}t + \frac{r^2 + a^2}{\Delta} \mathrm{d}r$$

and

$$\mathrm{d}\psi = \mathrm{d}\varphi + \mathrm{d}r^\# = \mathrm{d}\varphi + \frac{a}{\Delta} \mathrm{d}r$$

In the new coordinates, we thus have the relation

$$\mathrm{d}x^\mu \stackrel{*}{=} \begin{pmatrix} \mathrm{d}v \\ \mathrm{d}r \\ \mathrm{d}\theta \\ \mathrm{d}\psi \end{pmatrix} = \begin{pmatrix} \mathrm{d}t + \frac{r^2 + a^2}{\Delta} \mathrm{d}r \\ \mathrm{d}r \\ \mathrm{d}\theta \\ \mathrm{d}\varphi + \frac{a}{\Delta} \mathrm{d}r \end{pmatrix}$$

and thus generically the components of a vector x^μ in the old and new coordinate system are related by

$$x^v = x^t + \frac{r^2 + a^2}{\Delta} x^r \qquad x^\psi = x^\varphi + \frac{a}{\Delta} x^r$$

For the null vector l^μ this implies

$$l^v = l^t + \frac{r^2 + a^2}{\Delta} l^r = \frac{r^2 + a^2}{\Delta} - \frac{r^2 + a^2}{\Delta} = 0$$

and

$$l^\psi = l^\varphi + \frac{a}{\Delta} l^r = \frac{a}{\Delta} - \frac{a}{\Delta} = 0$$

so that in the new basis only the radial component of l^μ is nonvanishing and

$$l^\mu \stackrel{*}{=} -\delta^\mu_r$$

Although we have not proven it, the l^μ are in fact tangents to (lightlike) geodesics and in the new coordinates, they are at constant v, θ and ψ with an affine parameter r. The congruence of these “infalling” geodesics is called the principal null congruence. Let us now express the metric in the new coordinates. We find

$$
\begin{aligned}
\mathrm{d}\tau^2 &= \left(1-\frac{r_s r}{\rho^2}\right)\mathrm{d}t^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 - \frac{\Sigma}{\rho^2}\sin^2\theta\,\mathrm{d}\varphi^2 + \frac{2a\,r_s r}{\rho^2}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi \\
&= \left(1-\frac{r_s r}{\rho^2}\right)\left(\mathrm{d}v - \frac{r^2+a^2}{\Delta}\mathrm{d}r\right)^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 \\
&\quad -\frac{\Sigma}{\rho^2}\sin^2\theta\left(\mathrm{d}\psi - \frac{a}{\Delta}\mathrm{d}r\right)^2 + \frac{2a\,r_s r}{\rho^2}\sin^2\theta\left(\mathrm{d}v - \frac{r^2+a^2}{\Delta}\mathrm{d}r\right)\left(\mathrm{d}\psi - \frac{a}{\Delta}\mathrm{d}r\right) \\
&= \left(1-\frac{r_s r}{\rho^2}\right)\mathrm{d}v^2 - \left(2\left(1-\frac{r_s r}{\rho^2}\right)\frac{r^2+a^2}{\Delta} + \frac{2a\,r_s r}{\rho^2}\sin^2\theta\frac{a}{\Delta}\right)\mathrm{d}v\mathrm{d}r \\
&\quad +\left(\left(1-\frac{r_s r}{\rho^2}\right)\left(\frac{r^2+a^2}{\Delta}\right)^2 + \frac{2a\,r_s r}{\rho^2}\sin^2\theta\,\frac{r^2+a^2}{\Delta}\frac{a}{\Delta} - \frac{\rho^2}{\Delta} - \frac{\Sigma}{\rho^2}\sin^2\theta\frac{a^2}{\Delta^2}\right)\mathrm{d}r^2 \\
&\quad +\left(2\frac{\Sigma}{\rho^2}\sin^2\theta\frac{a}{\Delta} - \frac{2a\,r_s r}{\rho^2}\sin^2\theta\frac{r^2+a^2}{\Delta}\right)\mathrm{d}r\mathrm{d}\psi \\
&\quad +\frac{2a\,r_s r}{\rho^2}\sin^2\theta\,\mathrm{d}v\mathrm{d}\psi - \rho^2\mathrm{d}\theta^2 - \frac{\Sigma}{\rho^2}\sin^2\theta\,\mathrm{d}\psi^2
\end{aligned}
$$

Let us simplify this term by term. We find

$$
\begin{aligned}
g_{vr} &= -2\left(1-\frac{r_s r}{\rho^2}\right)\frac{r^2+a^2}{\Delta} - 2\frac{a\,r_s r}{\rho^2}\sin^2\theta\frac{a}{\Delta} \\
&= -2\frac{(\rho^2 - r_s r)(r^2+a^2) + r_s r\,\overbrace{a^2\sin^2\theta}^{r^2+a^2-\rho^2}}{\Delta\rho^2} \\
&= -2\frac{(\rho^2 - r_s r)(r^2+a^2) + r_s r\,(r^2+a^2-\rho^2)}{\Delta\rho^2} \\
&= -2\frac{\rho^2(r^2+a^2-r_s r)}{\Delta\rho^2} \\
&= -2
\end{aligned}
$$

$$
\begin{aligned}
g_{rr} &= \frac{(\rho^2 - r_s r)(r^2+a^2)^2 + 2a^2\,r_s r\sin^2\theta\,(r^2+a^2) - \Delta\rho^4}{\Delta^2\rho^2} - \frac{\Sigma}{\rho^2}\sin^2\theta\frac{a^2}{\Delta^2} \\
&= \frac{\rho^2(r^2+a^2)^2 + r_s r(r^2+a^2)(-r^2-a^2+2a^2\sin^2\theta\,) - \Delta\rho^4 - \Sigma a^2\sin^2\theta}{\Delta^2\rho^2} \\
&= \frac{\rho^2(r^2+a^2)^2 - (r^2+a^2-\Delta)(r^2+a^2)(\rho^2 - a^2\sin^2\theta\,) - \Delta\rho^4 - \Sigma a^2\sin^2\theta}{\Delta^2\rho^2} \\
&= \frac{\rho^2(r^2+a^2)^2 - (r^2+a^2-\Delta)(r^2+a^2)\rho^2 + ((r^2+a^2-\Delta)(r^2+a^2) - \Sigma)a^2\sin^2\theta - \Delta\rho^4}{\Delta^2\rho^2} \\
&= \frac{\Delta(r^2+a^2)\rho^2 + (-\Delta(r^2+a^2) + a^2\Delta\sin^2\theta)a^2\sin^2\theta - \Delta\rho^4}{\Delta^2\rho^2} \\
&= \frac{\Delta\overbrace{(r^2+a^2-\rho^2)}^{a^2\sin^2\theta}\rho^2 - \Delta\overbrace{(r^2+a^2\cos^2\theta)}^{\rho^2}a^2\sin^2\theta}{\Delta^2\rho^2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{r\psi} &= \frac{2a\sin^2\theta}{\Delta\rho^2}(\Sigma - r_s r(r^2+a^2)) \\
&= \frac{2a\sin^2\theta}{\Delta\rho^2}(\Sigma - (r^2+a^2-\Delta)(r^2+a^2)) \\
&= \frac{2a\sin^2\theta}{\Delta\rho^2}(-a^2\Delta\sin^2\theta + \Delta(r^2+a^2)) \\
&= \frac{2a\sin^2\theta}{\Delta\rho^2}\Delta\rho^2 \\
&= 2a\sin^2\theta
\end{aligned}
$$

So the metric has the form

$$d\tau^2 = \left(1 - \frac{r_s r}{\rho^2}\right)dv^2 - 2dvdr + 2a\sin^2\theta\, drd\psi + \frac{2ar_s r}{\rho^2}\sin^2\theta\, dvd\psi - \rho^2 d\theta^2 - \frac{\Sigma}{\rho^2}\sin^2\theta\, d\psi^2 \tag{37}$$

2.3.9 Kerr-Schild coordinates

One particularly interesting feature of the metric (37) is the way in which we can write its dependence on r_s. Rewriting

$$\begin{aligned}
g_{\psi\psi} &= -\frac{\Sigma}{\rho^2}\sin^2\theta \\
&= -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{r^2+a^2\cos^2\theta}\sin^2\theta \\
&= -\frac{(r^2+a^2)^2 - a^2(r^2+a^2-r_s r)\sin^2\theta}{r^2+a^2\cos^2\theta}\sin^2\theta \\
&= -\frac{(r^2+a^2)(r^2+a^2-a^2\sin^2\theta) + a^2 r_s r\sin^2\theta}{r^2+a^2\cos^2\theta}\sin^2\theta \\
&= -(r^2+a^2)\sin^2\theta - \frac{a^2 r_s r}{\rho^2}\sin^4\theta
\end{aligned}$$

the metric can be decomposed as

$$\begin{aligned}
d\tau^2 &= dv^2 - 2dvdr + 2a\sin^2\theta\, drd\psi - \rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta\, d\psi^2 \\
&\quad -\frac{r_s r}{\rho^2}dv^2 + \frac{2ar_s r}{\rho^2}\sin^2\theta\, dvd\psi - \frac{a^2 r_s r\sin^4\theta}{\rho^2}\, d\psi^2
\end{aligned}$$

On the right hand side, all terms in the first line are independent of r_s, while the second line is proportional to r_s. Pulling out the common factors, we notice that the remainder in fact is a complete square, i.e.

$$\begin{aligned}
d\tau^2 &= dv^2 - dvdr + 2a\sin^2\theta\, drd\psi - \rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta\, d\psi^2 \\
&\quad -\frac{r_s r}{\rho^2}(dv - a\sin^2\theta\, d\psi)^2
\end{aligned}$$

The second line on the right hand side can in fact be written in an even more compact form. Remembering the null vector $l^\mu \stackrel{*}{=} -\delta^\mu_r$, we find

$$\begin{aligned}
l_\mu dx^\mu &= l^\mu g_{\mu\nu}\, dx^\nu \\
&= -g_{r\nu}\, dx^\nu \\
&= -g_{rv}dv - g_{r\psi}\, d\psi \\
&= dv - a\sin^2\theta\, d\psi
\end{aligned}$$

so in fact the metric is

$$\begin{aligned}
d\tau^2 &= dv^2 - 2dvdr + 2a\sin^2\theta\, drd\psi - \rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta\, d\psi^2 \\
&\quad -\frac{r_s r}{\rho^2}(l_\mu dx^\mu)^2
\end{aligned}$$

It is interesting that in the limit $r_s \to 0$ only the first line of the metric survives. But we already know that for $r_s \to 0$, the Kerr metric describes flat Minkovski spacetime in a peculiar coordinate system. If we denote the flat metric as $\eta_{\mu\nu}$ we can ultimately write

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{r_s r}{\rho^2} l_\mu l_\nu$$

To better understand the metric, we now make the coordinate transformation

$$x + iy = (r + ia)\sin\theta e^{i\psi} \qquad z = r\cos\theta \qquad t' = v - r$$

so

$$\mathrm{d}x + i\mathrm{d}y = \mathrm{d}r\sin\theta e^{i\psi} - (a - ir)\sin\theta e^{i\psi}\,\mathrm{d}\psi + (r + ia)\cos\theta e^{i\psi}\,\mathrm{d}\theta$$

and

$$\mathrm{d}z = \mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta \qquad \mathrm{d}t' = \mathrm{d}v - \mathrm{d}r$$

This implies that

$$\begin{aligned}
\mathrm{d}x^2 + \mathrm{d}y^2 &= (\mathrm{d}x + i\mathrm{d}y)(\mathrm{d}x - i\mathrm{d}y) \\
&= (\mathrm{d}r\sin\theta - (a - ir)\sin\theta\mathrm{d}\psi + (r + ia)\cos\theta\mathrm{d}\theta) \\
&\quad (\mathrm{d}r\sin\theta - (a + ir)\sin\theta\,\mathrm{d}\psi + (r - ia)\cos\theta\,\mathrm{d}\theta) \\
&= (\sin\theta(\mathrm{d}r - a\mathrm{d}\psi) + r\cos\theta\mathrm{d}\theta)^2 + (r\sin\theta\mathrm{d}\psi + a\cos\theta\mathrm{d}\theta)^2 \\
&= \sin^2\theta\mathrm{d}r^2 + (r^2 + a^2)\cos^2\theta\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\mathrm{d}\psi^2 \\
&\quad -2a\sin^2\theta\mathrm{d}r\,\mathrm{d}\psi + 2r\sin\theta\cos\theta\mathrm{d}r\,\mathrm{d}\theta
\end{aligned}$$

and thus

$$\begin{aligned}
\mathrm{d}t'^2 - \mathrm{d}x^i\mathrm{d}x^i &= (\mathrm{d}v - \mathrm{d}r)^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - (\mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta)^2 \\
&= \mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r + \mathrm{d}r^2 - \mathrm{d}r^2\cos^2\theta - r^2\sin^2\theta\mathrm{d}\theta^2 + 2r\sin\theta\cos\theta\mathrm{d}r\mathrm{d}\theta \\
&\quad -\sin^2\theta\mathrm{d}r^2 - (r^2 + a^2)\cos^2\theta\mathrm{d}\theta^2 - (r^2 + a^2)\sin^2\theta\mathrm{d}\psi^2 \\
&\quad +2a\sin^2\theta\mathrm{d}r\,\mathrm{d}\psi - 2r\sin\theta\cos\theta\mathrm{d}r\,\mathrm{d}\theta \\
&= \mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r - (r^2 + a^2\cos^2\theta)\,\mathrm{d}\theta^2 - (r^2 + a^2)\sin^2\theta\mathrm{d}\psi^2 + 2a\sin^2\theta\mathrm{d}r\,\mathrm{d}\psi \\
&= \mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r - \rho^2\,\mathrm{d}\theta^2 - (r^2 + a^2)\sin^2\theta\mathrm{d}\psi^2 + 2a\sin^2\theta\mathrm{d}r\,\mathrm{d}\psi
\end{aligned}$$

which is exactly the flat part of the Kerr metric. The coordinates t', x, y and z are thus the time and carthesian coordinates in the case of flat Minkovski space. They are called Kerr-Schild coordinates.

2.3.10 A closer look at the singularity

We can now finally come back to the question of the singularity that occurs for $r^2 + a^2\cos^2\theta = 0$. In Boyer-Lindquist coordinates this had the strange implication that the singularity occurs only for $r = 0$ in the equatorial plane. In Kerr-Schild coordinates we see that

$$x^2 + y^2 = (r^2 + a^2)\sin^2\theta \qquad z = r\cos\theta$$

so $r = 0$ corresponds to

$$x^2 + y^2 = a^2\sin^2\theta \qquad z = 0$$

which is not a point but rather a disc with radius r. The inside of this disc corresponds to values $\sin^2\theta < 1$ and at its boundary we have $\sin^2\theta = 1$, which is the location of the singularity. The singularity is thus annular in shape and in Kerr-Schild coordinates corresponds to the circle

$$x^2 + y^2 = a^2 \qquad z = 0$$

In fact, if we stay clear of the equatorial plane, we can not hit the singularity. We can imagine a trajectory, e.g. with $\cos\theta = 1$ that passes right through the middle of the ring and exits again on the other side.

2.3.11 Going through the singularity

Let us try to see what happens if we aim for passing through the annular singularity of a Kerr black hole. Going in we will first pass through two horizons, r_+ and then r_- before we even reach $r = 0$. This may sound simple enough, but one has to remember that already at the time when we approach the first horizon, there will be an infinite time dilation and thus we will see the entire timelike future of the outside region when crossing the horizon. This begs the question of where we actually emerge - will t be an asymptotically flat region or will we be trapped inside the black hole somehow?

In order to answer this question, we will need to explore the global structure of the Kerr metric, just as we did for the Schwarzschild metric. In the case of the Schwarzschild black hole, we introduced the lightlike Kruskal-Szekeres coordinates $v = t + r^*$ and $u = t - r^*$ for that purpose. When we do the same for the Kerr metric, we find

$$\mathrm{d}v = \mathrm{d}t + \frac{r^2 + a^2}{\Delta}\mathrm{d}r \qquad \mathrm{d}u = \mathrm{d}t - \frac{r^2 + a^2}{\Delta}\mathrm{d}r$$

so

$$\mathrm{d}t = \frac{\mathrm{d}u + \mathrm{d}v}{2} \qquad \mathrm{d}r = \frac{\Delta(\mathrm{d}v - \mathrm{d}u)}{2(r^2 + a^2)}$$

which we can plug into the Boyer-Lindquist form of the Kerr metric. In order to simplify things, we will only look at the rotation axis, i.e. $\theta = 0$. This should be sufficient for our purposes as it corresponds to going right through the middle of the singularity. We can rewrite the metric (28) as

$$\begin{aligned}
\mathrm{d}\tau^2 &= \left(1 - \frac{r_s r}{r^2 + a^2}\right)\mathrm{d}t^2 - \frac{r^2 + a^2}{r^2 + a^2 - r_s r}\mathrm{d}r^2 \\
&= \frac{r^2 + a^2 - r_s r}{r^2 + a^2}\left(\frac{\mathrm{d}u + \mathrm{d}v}{2}\right)^2 - \frac{r^2 + a^2}{r^2 + a^2 - r_s r}\left(\frac{(r^2 + a^2 - r_s r)(\mathrm{d}v - \mathrm{d}u)}{2(r^2 + a^2)}\right)^2 \\
&= \frac{1}{4}\frac{r^2 + a^2 - r_s r}{r^2 + a^2}((\mathrm{d}u + \mathrm{d}v)^2 - (\mathrm{d}u - \mathrm{d}v)^2) \\
&= \frac{r^2 + a^2 - r_s r}{r^2 + a^2}\mathrm{d}u\mathrm{d}v \\
&= \frac{(r - r_+)(r - r_-)}{r^2 + a^2}\mathrm{d}u\mathrm{d}v
\end{aligned}$$

Let us now see which coordinates u und v the horizon at r_+ corresponds to. Remembering that r^* is given by

$$r^* = r + \frac{r_s r_+}{\sqrt{r_s^2 - 4a^2}}\ln\left|\frac{r - r_+}{r_+}\right| - \frac{r_s r_-}{\sqrt{r_s^2 - 4a^2}}\ln\left|\frac{r - r_-}{r_-}\right| \tag{38}$$

we see that

$$r^* \xrightarrow{r\to r_+} -\infty \qquad u \xrightarrow{r\to r_+} \infty \qquad v \xrightarrow{r\to r_+} -\infty$$

In these coordinates the metric thus only describes the outside region of the Kerr black hole $r > r_+$. In that outside region we can as well write

$$r^* = r + \frac{r_s r_+}{\sqrt{r_s^2 - 4a^2}} \ln\frac{r - r_+}{r_+} - \frac{r_s r_-}{\sqrt{r_s^2 - 4a^2}} \ln\frac{r - r_-}{r_-}$$

which allows for a smoother continuation. Similar to the Schwarzschild case (22), we now introduce coordinates that are supposed to eliminate this logarithmic divergence. We define

$$U_+ = -e^{-\kappa_+ u} \qquad V_+ = e^{\kappa_+ v}$$

with a constant κ_+ yet to be determined. The differentials now read

$$\mathrm{d}U_+ = -\kappa_+ U_+ \mathrm{d}u \qquad \mathrm{d}V_+ = \kappa_+ V_+ \mathrm{d}v$$

so the metric can be rewritten as

$$\mathrm{d}\tau^2 = -\frac{(r - r_+)(r - r_-)}{r^2 + a^2} \frac{1}{\kappa_+^2 U_+ V_+} \mathrm{d}U_+ \mathrm{d}V_+ \tag{39}$$

We can express the new coordinates as

$$\begin{aligned} U_+ &= -e^{-\kappa_+(t - r^*)} \\ &= -e^{-\kappa_+(t - r)} \left(\frac{r - r_+}{r_+}\right)^{\kappa_+ \frac{r_s r_+}{\sqrt{r_s^2 - 4a^2}}} \left(\frac{r - r_-}{r_-}\right)^{-\kappa_+ \frac{r_s r_-}{\sqrt{r_s^2 - 4a^2}}} \\ V_+ &= e^{\kappa_+(t + r^*)} \\ &= e^{\kappa_+(t + r)} \left(\frac{r - r_+}{r_+}\right)^{\kappa_+ \frac{r_s r_+}{\sqrt{r_s^2 - 4a^2}}} \left(\frac{r - r_-}{r_-}\right)^{-\kappa_+ \frac{r_s r_-}{\sqrt{r_s^2 - 4a^2}}} \end{aligned}$$

so that

$$U_+ V_+ = -e^{2\kappa_+ r} \left(\frac{r - r_+}{r_+}\right)^{2\kappa_+ \frac{r_s r_+}{\sqrt{r_s^2 - 4a^2}}} \left(\frac{r - r_-}{r_-}\right)^{-2\kappa_+ \frac{r_s r_-}{\sqrt{r_s^2 - 4a^2}}}$$

If we now choose

$$\kappa_+ = \frac{\sqrt{r_s^2 - 4a^2}}{2 r_s r_+} = \frac{r_+ - r_-}{2 r_+ (r_+ + r_-)}$$

and define a similar

$$\kappa_- = \frac{\sqrt{r_s^2 - 4a^2}}{2 r_s r_-} = \frac{r_+ - r_-}{2 r_- (r_+ + r_-)}$$

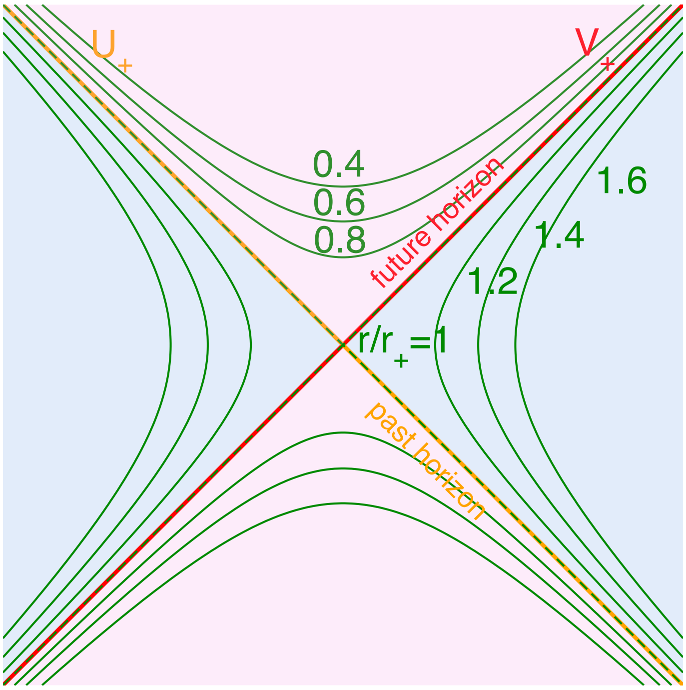
we obtain

$$U_+ V_+ = -e^{2\kappa_+ r} \frac{r - r_+}{r_+} \left(\frac{r - r_-}{r_-}\right)^{-\frac{\kappa_+}{\kappa_-}} \tag{40}$$

The metric (39) may thus be written as

$$d\tau^2 = e^{-2\kappa_+ r} \frac{r_+(r-r_-)}{r^2+a^2} \left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{\kappa_-}} \frac{1}{\kappa_+^2} dU_+ dV_+$$

which is regular at the outer horizon r_+. According to (40), the outer horizon is at $U_+V_+=0$. Curves of constant $r > r_+$ are hyperbolae in the coordinates U_+ and V_+ with $U_+V_+ < 1$, while curves of constant $r_+ > r > r_-$ are hyperbolae with $U_+V_+ > 1$. The following diagram depicts this for a Kerr black hole with $a = 0.4 r_s$.



This is all very similar to the Schwarzschild case. The only real difference is that with this coordinate patch we can only describe the geometry for $r > r_-$. According to (40), U_+V_+ diverges as $r \to r_-$, so we have to use different coordinates as we cross r_-. These coordinates are in fact easy to find. We define

$$U_- = -e^{\kappa_- u} \qquad V_- = -e^{-\kappa_- v}$$

so

$$dU_- = \kappa_- U_- du \qquad dV_- = -\kappa_- V_- dv$$

and the metric can be written as

$$d\tau^2 = -\frac{(r-r_+)(r-r_-)}{r^2+a^2} \frac{1}{\kappa_-^2 U_- V_-} dU_- dV_- \tag{41}$$

With this new coordinate patch we can only describe the region $r < r_+$, starting from $r > r_-$. In that region, we can write (38) as

$$r^* = r + \frac{r_s r_+}{\sqrt{r_s^2 - 4a^2}} \ln \frac{r_+ - r}{r_+} - \frac{r_s r_-}{\sqrt{r_s^2 - 4a^2}} \ln \frac{r - r_-}{r_-}$$

In terms of our original r and t these new coordinates are thus

$$\begin{aligned} U_- &= -e^{\kappa_-(t-r^*)} \\ &= -e^{\kappa_-(t-r)}\left(\frac{r_+-r}{r_+}\right)^{-\frac{\kappa_-}{2k_+}}\sqrt{\frac{r-r_-}{r_-}} \\ V_- &= -e^{-\kappa_-(t+r^*)} \\ &= -e^{-\kappa_+(t+r)}\left(\frac{r_+-r}{r_+}\right)^{-\frac{\kappa_-}{2\kappa_+}}\sqrt{\frac{r-r_-}{r_-}} \end{aligned} \tag{42}$$

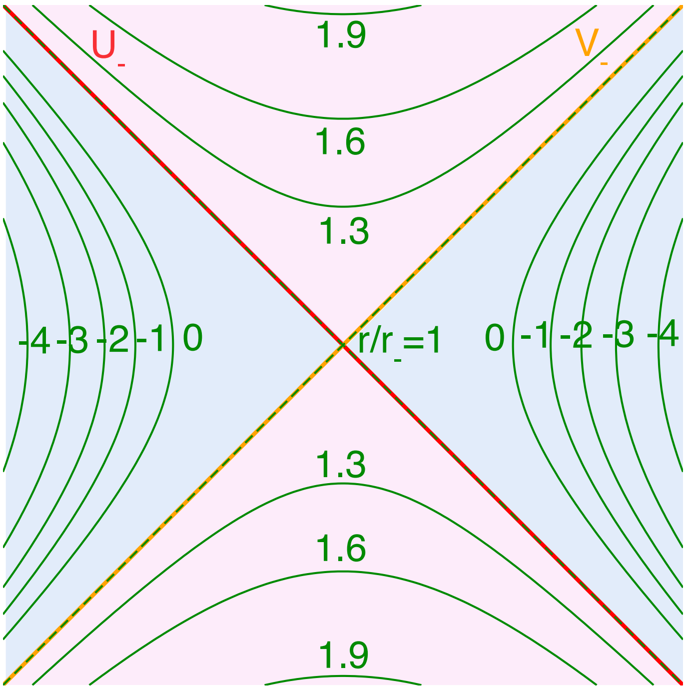
so that

$$U_-V_- = e^{-2\kappa_- r}\,\frac{r-r_-}{r_-}\left(\frac{r_+-r}{r_+}\right)^{-\frac{\kappa_-}{k_+}} \tag{43}$$

The metric (41) in these new coordinates reads

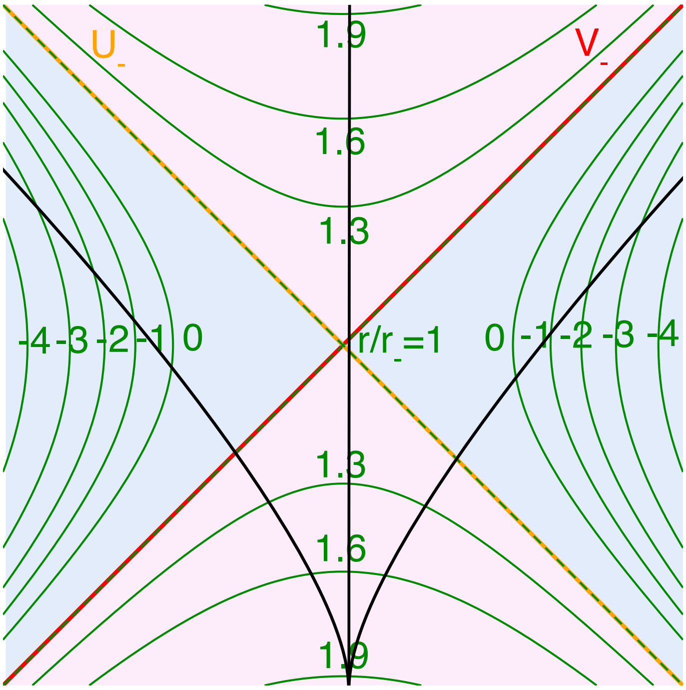
$$\mathrm{d}\tau^2 = e^{2\kappa_- r}\frac{(r_+-r)r_-}{r^2+a^2}\,\frac{1}{\kappa_-^2}\left(\frac{r_+-r}{r_+}\right)^{\frac{\kappa_-}{k_+}}\mathrm{d}U_-\mathrm{d}V_-$$

which is regular around r_-. According to (43), the horizon occurs at $U_-V_-=0$ and curves of conatant r are hyperbolae with a constant U_-V_-. For our example case $a=0.4r_s$, this looks as follows:

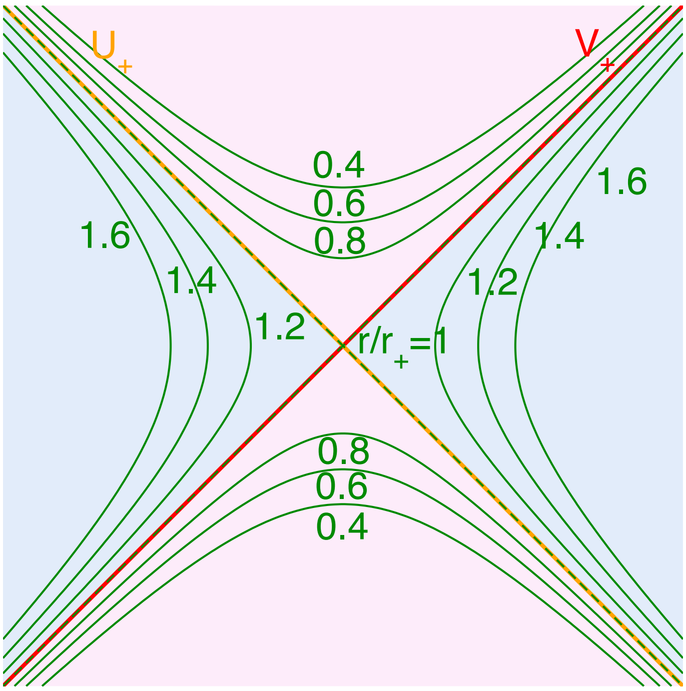


First we note, that curves of constant $r_+ > r > r_-$ are spacelike as required (because they were spacelike in the other coordinate patch). The position of the outer horizon is both at the bottom and at the top at an infinite coordinate distance. When we pass inside the inner horizon $r < r_-$ however, they become timelike again like they were outside the outer horizon. Thus the $r=0$ line is timelike again, in contrast to the Schwarzschild case. Also in contrast to the Schwarzschild case, $r=0$ is not singular outside the equatorial plane. Since we are outside the equatorial plane, we may just continue our spacetime to negative values of r. In fact, the coordinate mappings (42) are regular for all negative values of r and we find two asymptotic regions as $r \to -\infty$, corresponding to the left and right quadrant.

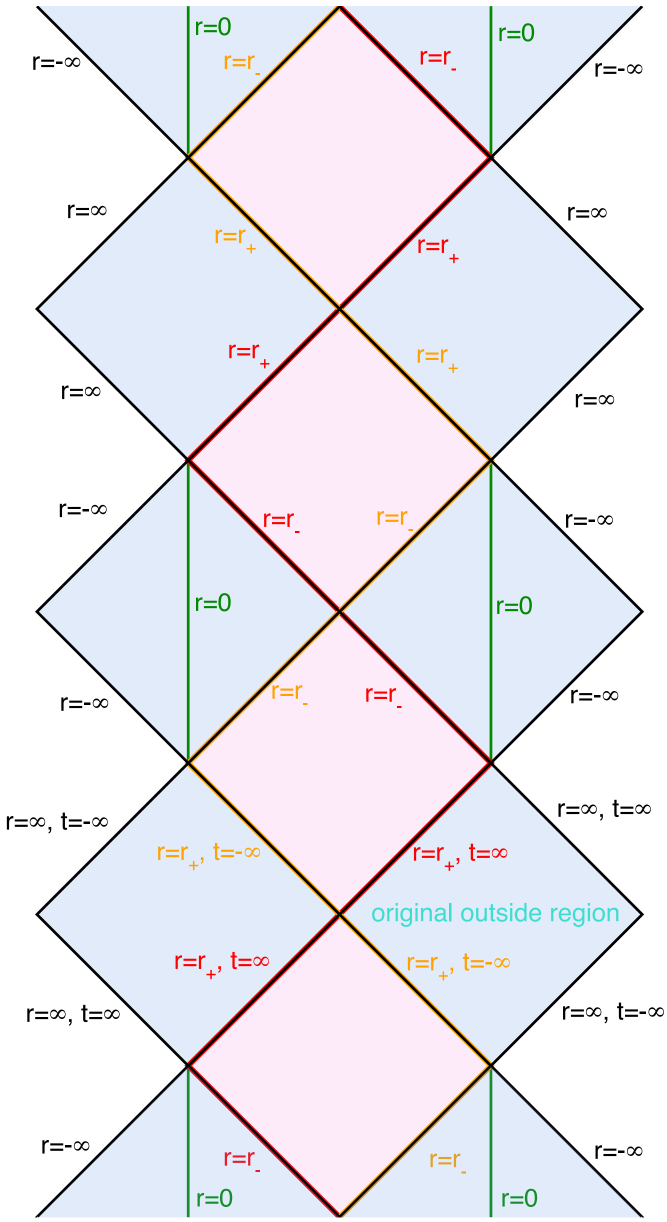
Now imagine that we have crossed the outer horizon at r_+, entering the diagram from the lower quadrant. Obviously we can choose future directed world lines which go to either of the $r \to -\infty$ asymptotic regions or into the upper quadrant where we again proceed to r_+



If we follow the world line into the region where r is increasing again, our coordinates are again not sufficient as we approach r_+. But now we know how to handle this: we just use another Kruskal patch with U_+ and V_+ coordinates. Let us depict this patch again:



Remember that we have just crossed the inner horizon r_- on a trajectory with increasing r. We thus enter the new coordinate patch not from the right quadrant but from the bottom one. Consequently, we may actually proceed to all three quadrants of this new patch. At this point it is important to note that although we are using the same coordinates here than in our first Kruskal patch (where we started outside the outer horizon), the regions are nonetheless not identical. The first U_+-V_+ covers a region in the past of the U_--V_- patch, while the second one covers a region in its future. Remembering that spacetime is a manifold, there is nothing particularly surprising about it. It is simply an instance of a manifold where one single coordinate patch is not sufficient to cover the entire manifold. In fact, our extension of the manifold is not even complete. The past of our first U_+-V_+ patch as well as the future of our second still end at r_-. We can extend both by adding another U_--V_- patch, which will make them end in r_+, requiring yet two more U_+-V_+ patches and so on. The maximal extension of the Kerr metric along its axis thus consists of an infinite number of alternating U_+-V_+ and U_--V_- patches with the entire outside (of r_+) world represented by a single right quadrant of one U_+-V_+ patch. We can depict this compactly in a Penrose diagram:



Coming back to our original question of passing through the annular singularity, we see that this will bring us to an $r<0$ patch. Somewhat surprisingly though this does not seem to be the only option. There are timelike curves that emerge into the asymptotic region of another U_+-V_+ patch and there are those which bounce between r_+ and r_-.

2.3.12 The inner horizon

We have seen that the maximally extended Kerr metric has some very peculiar features. It seems that an observer may move into an entirely different universe. Some of these universes look like copies of the original one, but there are also the coordinate patches with $r < 0$. The obvious question is whether in a physically realistic black hole these additional regions are accessible or whether they are mere mathematical curiosities. But before we try to answer this question, let us take a closer look at the $r < 0$ region. The Kerr metric in Boyer-Lindquist coordinates (28) shows us that r appears only in terms r^2 and $r_s r$. The r^2 terms are unaffected by a sign flip in r while for the $r_s r$ terms the sign flip in r may in fact be shifted to a sign flip in r_s. We can thus conclude that observers in the $r < 0$ region can in fact interpret their situation as being in a Kerr metric of negative r_s, i.e. of negative mass. This is problematic, since there are no horizons for $r_s < 0$ - the equation $\Delta = r^2 - r_s r + a^2 = 0$ has no positive, real solutions for $r_s < 0$. Thus the singularity at $r = 0$, $\theta = \pi/2$ is not shielded in this region, but exposed as a naked singularity. So it seems that if a tunnel to other universes is indeed open, the cosmic censorship conjecture is violated.

In order to see what is actually happening, we investigate the behaviour of a radially infalling observer along the symmetry axis. Along the $\theta = 0$ axis (using the identity $a^2 = r_+ r_-$), we can write the Kerr metric as

$$\mathrm{d}\tau^2 = \frac{(r - r_+)(r - r_-)}{r^2 + r_+ r_-}\mathrm{d}t^2 - \frac{r^2 + r_+ r_-}{(r - r_+)(r - r_-)}\mathrm{d}r^2$$

The energy per mass along a timelike geodesic at $\theta = 0$ is thus given by

$$\begin{aligned} e &= g_{t\mu} u^\mu \\ &= \frac{(r - r_+)(r - r_-)}{r^2 + r_+ r_-}\dot{t} \end{aligned}$$

The normalization condition of the eigenvector gives us

$$\begin{aligned} 1 &= \frac{(r - r_+)(r - r_-)}{r^2 + r_+ r_-}\dot{t}^2 - \frac{r^2 + r_+ r_-}{(r - r_+)(r - r_-)}\dot{r}^2 \\ &= \frac{r^2 + r_+ r_-}{(r - r_+)(r - r_-)}(e^2 - \dot{r}^2) \end{aligned}$$

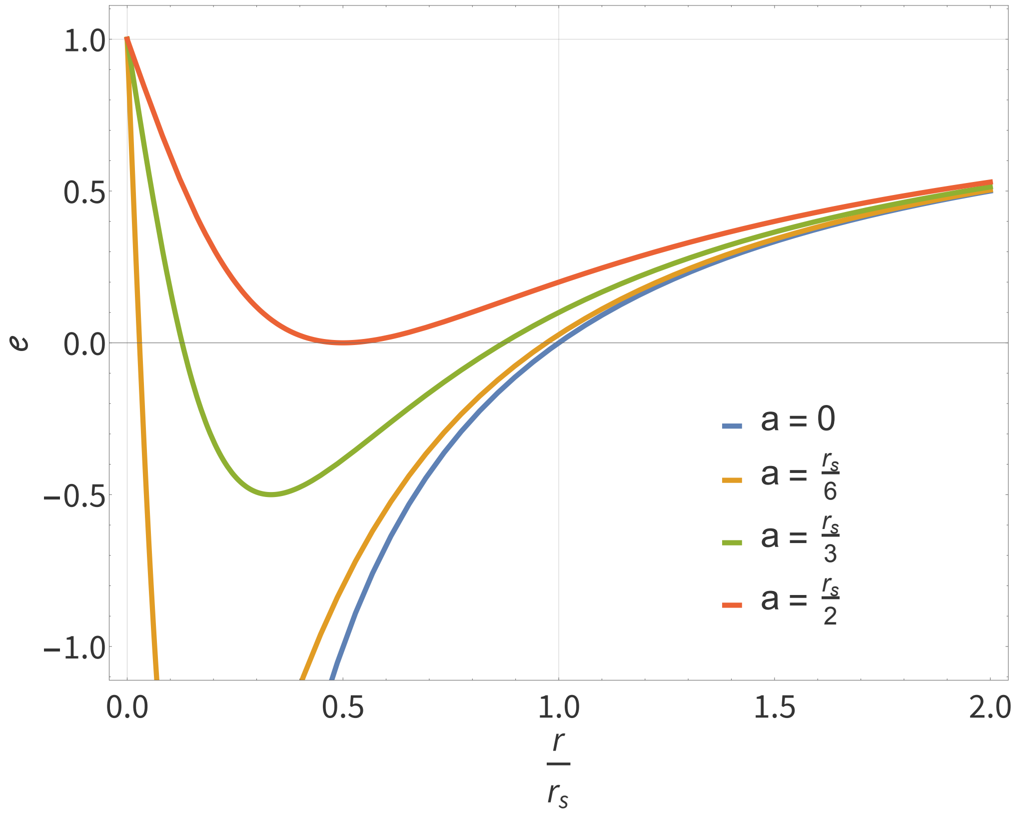
so we have the geodesic equation

$$e^2 = \dot{r}^2 + \frac{(r - r_+)(r - r_-)}{r^2 + r_+ r_-}$$

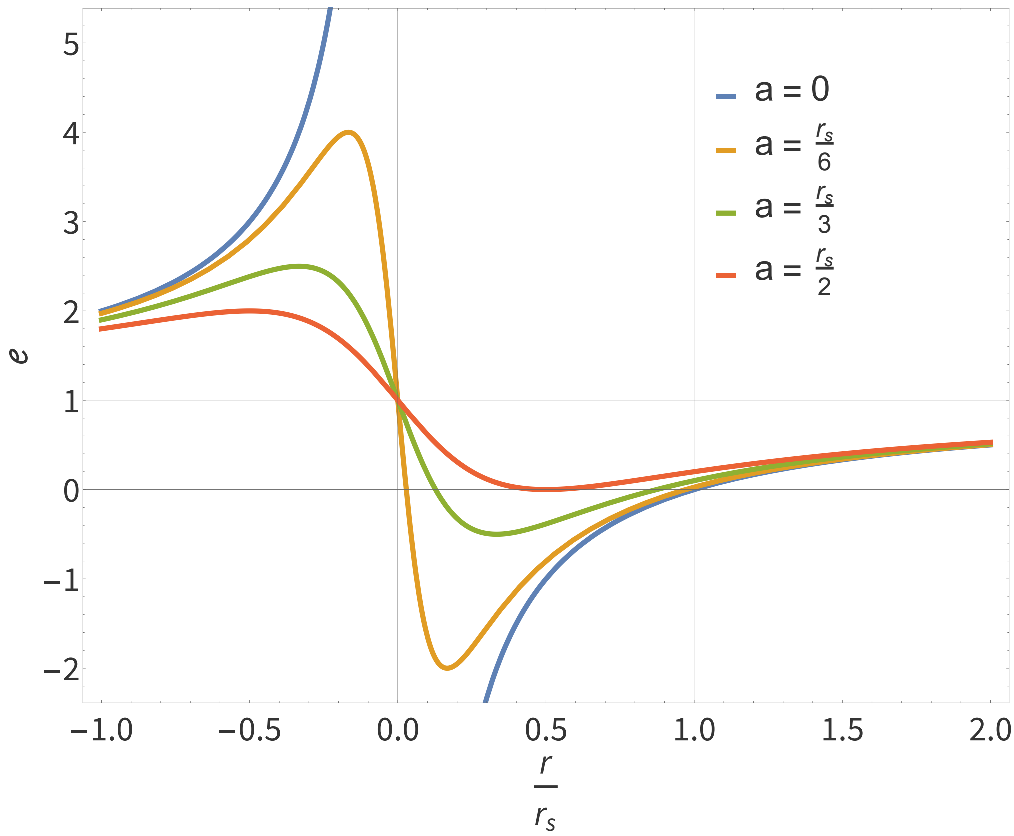
This is equivalent to the motion of a classical particle in the coordinate r in an effective potential, which is given by

$$U_{\mathrm{eff}} = \frac{(r - r_+)(r - r_-)}{r^2 + r_+ r_-}$$

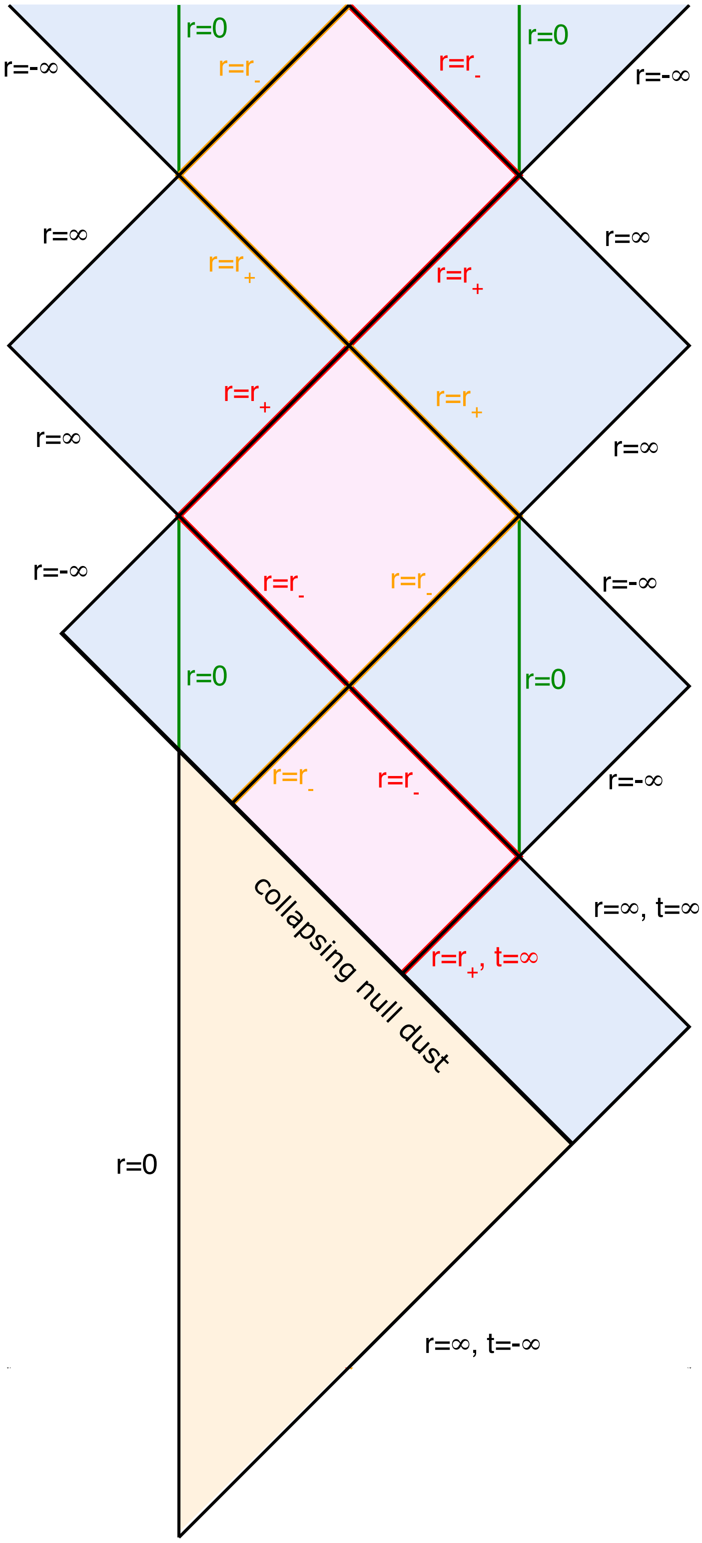
which, for various values of a looks like this:



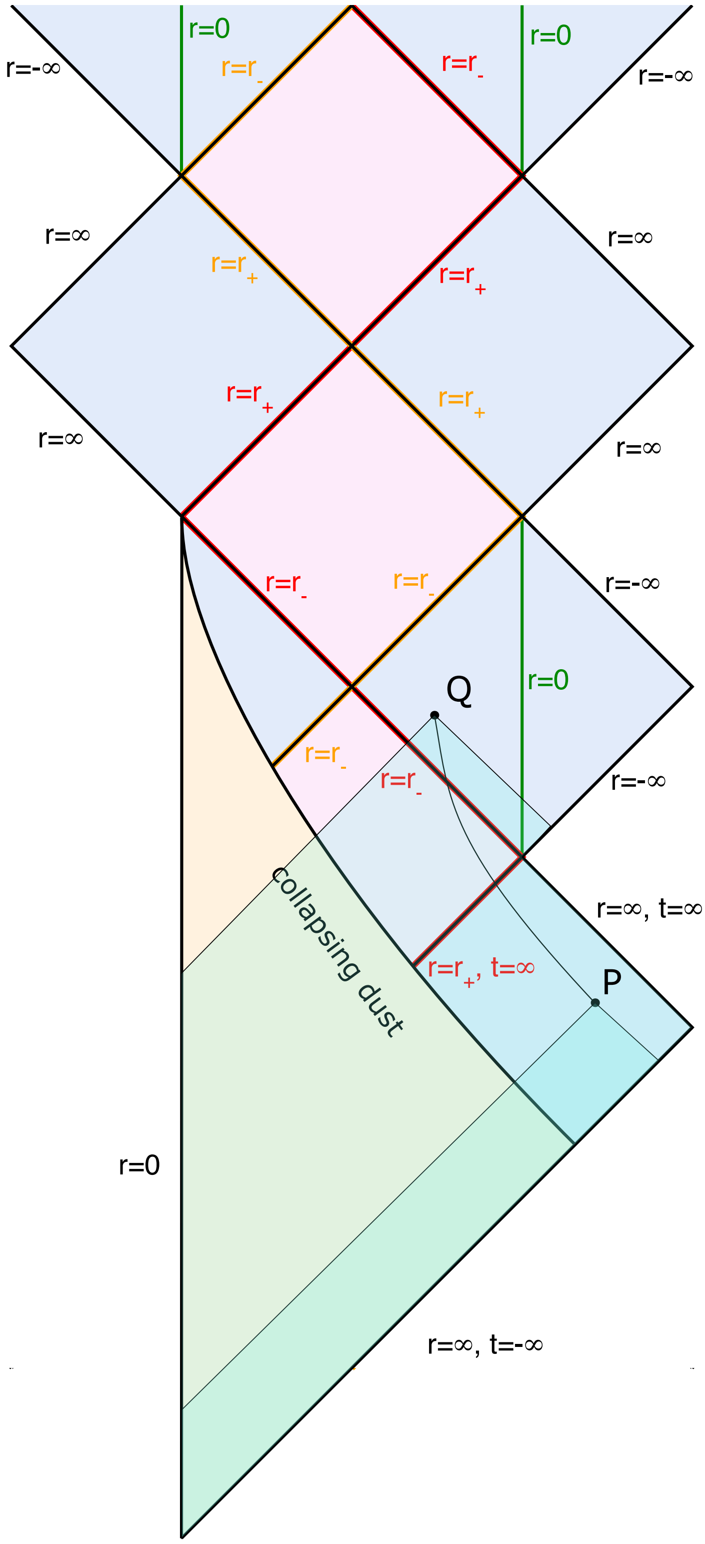
Interestingly, the effective potential actually has a minimum for all $a > 0$. It lies between the two horizons, which are just the zero crossings of the potential. Objects with $e < 1$ thus oscillate radially and objects with $e < 0$ do this in between the two horizons. More generally, the inner horizon seems to be repulsive, which is remarkable in itself. With $e > 1$, we can actually overcome this repulsion to enter the region $r < 0$. In order to see what happens beyond the crossing of $r = 0$, we can plot the effective potential in that region, too:



As we can see, the effective potential has a finite maximum for all $a > 0$ and we can enter the asymptotic regime $r \to -\infty$ with finite energy. There the gravitational force is repulsive, consistent with the fact that observers in this region assign a negative mass to the black hole. So for a perfect Kerr black hole all those additional regions are accessible, but the question is, whether they are realized in nature. Since the detailed calculations are quite lengthy, we will argue only qualitatively, mostly with Penrose diagrams. Let us begin by seeing how a gravitational collapse may produce a Kerr black hole. For the Schwarzschild black hole, we had the Vaidya metric, which was in fact an exact solution for a certain form of ingoing null dust (see exercises). This solution can in fact be extended to the Kerr case, but there it violates an energy condition. Nonetheless, when we draw the Penrose diagram of an ideal collapse of a thin shell of null dust, we can see the maximum possible structure a Kerr black hole can carry if we assume that it was somehow produced from normal matter. We just glue together a Minkovski metric in the inside region with a metric with a Kerr metric in the outside region:



This form of the collapse assumes that the null dust actually collapses right through $r = 0$ and into the asymptotic region $r \to -\infty$. This is rather clearly unphysical and a more physical collapse scenario might look as follows:



It now becomes clear that whenever we want to enter one of the "exotic" regions, we have to cross the inner horizon r_-. Now assume that we start in the outside region, at point P and follow a timelike trajectory to point Q. As we pass the inner horizon r_-, two distinct things happen: First, the entire outside region of the original universe we lived in is inside the past light cone. This implies that every singal that was ever sent from this region towards the black hole reaches us in one instant and, because we are at the inner horizon, all of it is infinitely blueshifted. This leads to the generic belief that the inner horizon is unstable, as a small perturbation from outside has an infinite effect there. The second thing that happens at r_- is the sudden appearance of an entirely different universe in our past. Up to the crossing of r_-, the past light cone of any observer was inside the original universe and thus its future is determined by it alone (this is indicated in the figure for point P). Once we cross r_-, this is no more so. As seen in the figure, the past light cone of point Q includes the past boundary of the new patch at $r = -\infty$. Although this is not a singularity, this is a pathological behaviour. We have some geodesics that are not extensible into the past beyond the end of this new patch of the universe. This suggests that a Kerr black hole that forms from e.g. stellar collapse will have a different internal structure than the maximal extension and passage to other patches are probably not possible in the real world.