Grundlagen der Elementar- und Astroteilchenphysik

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- Pierre Ramond: **Field theory: a modern primer**
  Chapter 1.2 – 1.7 (Lorentz and Poincare Symmetry, Spinors)
- Ta-Pei Cheng, Ling-Fong Li: **Gauge theory of elementary particle physics**
  Chapter 1.1 - 1.3 (Quantum fields)
  Chapter 4.1,4.2 (Lie Groups)
  Chapter 5.1, 5.3 (Global Symmetries, Goldstone theorem)
  Chapter 8.1 (Local Symmetries)
  Chapter 8.3 (Higgs mechanism)
  Chapter 11.1-11.3 (structure of the Standard Model)
  Chapter 12 (Standard Model phenomenology)
- Further reading
  L. B. Okun: **Leptons and Quarks**
  Chapter 3. (Muon decay)
  Chapter 29. (Appendix)
  K. Huang: **Quarks Leptons & Gauge fields**
  Chapter 3. (Electrodynamics)
  Chapter 4. (Non-abelian gauge theories)
  Chapter 6. (Standard model)
  Chapter 7-8. (Quantization)

Special relativity postulates that light travels with the same speed in all reference frames. Thus, in two frames

$$c^2t^2 - x_i x_i = c^2 t'^2 - x'_i x'_i \equiv s^2 \tag{1}$$

is equal. Since c is just a constant, $c = 299792458\ \mathrm{m/s}$, it simply defines the relation between metre and second. We will use $c \equiv 1$ units.
For a compact notation we introduce the $g_{\mu\nu}$ metric tensor

$$s^2 = x^0 x^0 - x^i x^i \equiv x^\mu x^\nu g_{\mu\nu} \tag{2}$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3}$$

Here x^μ is a four vector where space and time coordinates are packed together $(t, \vec{x})$.

When we change between frames (rotation or boost) the coordinates transform linearly.

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} = \Lambda^{\mu}_{0} x^{0} + \Lambda^{\mu}_{i} x^{i} \tag{4}$$

The transformation matrix Λ is a *Lorentz transformation* if $x^{\mu} x^{\nu} g_{\mu\nu}$ is invariant:

$$g_{\mu\nu} x'^{\mu} x'^{\nu} = g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} x^{\rho} x^{\sigma} = g_{\rho\sigma} x^{\rho} x^{\sigma} \tag{5}$$

it holds for any x^{μ} if

$$g_{\rho\sigma} = g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} \tag{6}$$

or, writing in matrix form:

$$\Lambda^{T} g \Lambda = g \tag{7}$$

Taking its determinant

$$\det g = \det \Lambda^{T} \det g \det \Lambda \tag{8}$$

we can deduce that

$$\det \Lambda = \pm 1 \tag{9}$$

$+1$: proper Lorentz tranformations (rotation, boost),
-1: improper (e.g. space inversion: $x^0 \to x^0$, $x^i \to -x^i$, perhaps combined with a proper transformation)

- *a) Rotations*
  If R is a 3-by-3 rotation matrix ($R^{+} = R^{-1}$), then

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \overset{for\ \ example}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (10)$$

  the determinant is $\det\Lambda = \det R = \pm 1$. (Space inversion or reflection is an improper transformation.)

- *b) Boosts* If the boost occurs in the 1st space direction:

$$\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (11)$$

  The determinant: $\det\Lambda = \cosh^2\eta - \sinh^2\eta = 1$.
  We can write the components in a more familiar form:

$$\cosh\eta = \frac{1}{\sqrt{1-v^2}} \qquad \sinh\eta = \frac{v}{\sqrt{1-v^2}} \qquad (12)$$

Time and space coordinates can be packed into a four-vector v^μ. A four-vector is defined with the feature that by changing the reference frame a well defined transformation rule appiles: the Lorentz transformation.

$$v^\mu - (v')^\mu = \Lambda^\mu{}_\nu v^\nu \tag{13}$$

One can define as scalar product between four-vectors, for this to work we need to introduce the metric tensor g:

$$v \cdot w = v^\mu g_{\mu\nu} w^\nu = \sum_{\mu,\nu=0}^{4} v^\mu g_{\mu\nu} w^\nu \tag{14}$$

with

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

Then $v \cdot w$ is a scalar, indeed: it remains unchanged after a Lorentz transformation.

$$v' \cdot w' = \Lambda^\mu\ \rho v^\rho g_{\mu\nu} \Lambda^\nu\ \sigma w^\sigma = v^\mu g_{\mu\nu} w^\nu = v \cdot w \tag{16}$$

We used the defining equation of the Lorentz transformation $\Lambda^T g \Lambda = g$.

The most important example for a tensor with two indices is the (antisymmetric) electromagnetic field strength tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (17)$$

Ignoring the space-time dependence, the transformation rule reads: $(F')^{\mu\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\rho}F^{\rho\sigma}$ Can one build a scalar from this?

- $F^{\mu\nu}g_{\mu\nu} = 0$, too trivial
- $\frac{1}{4}F^{\mu\nu}F^{\rho\sigma}g_{\mu\rho}g_{\nu\sigma} = -(\vec{E}^2 - \vec{B}^2)$ this is the action
- Is there a relativistic version for $\vec{E}\cdot\vec{B}$? This product is a pseudo scalar, actually: it changes sign under reflection.

$$\vec{E} \to -\vec{E} \quad (18)$$

$$\vec{B} \to \vec{B} \quad (19)$$

$$\vec{E}\cdot\vec{B} \to -\vec{E}\cdot\vec{B} \quad (20)$$

$$\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B} \to \vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B} \quad (21)$$

Let's introduce the 4 dimensional *Levi-Civita tensor* $\epsilon_{\mu\nu\rho\sigma}$

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon_{\nu\mu\rho\sigma} = -\epsilon_{\mu\nu\sigma\rho} = -\epsilon_{\sigma\nu\rho\mu} \tag{22}$$

$$\epsilon_{0123} = 1 \tag{23}$$

This defines the *dual field strength tensor*:

$$\tilde{F}^{\mu\nu} = \frac{1}{2} g^{\mu\alpha} g^{\nu\beta} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{24}$$

Now we construct a pseudo scalar: $F \cdot \tilde{F}$

$$(F^{\mu\nu} g_{\mu\rho} g_{\nu\sigma}) \tilde{F}^{\rho\sigma} = -\mathrm{Tr} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

Working out product we get $F \cdot \tilde{F} = -\frac{1}{4} \vec{E}\vec{B}$.

Until now all tensors had upper *(contravariant)* indices, e.g. $F^{\mu\nu}$, u^μ.
Using the metric tensor we introduce the *covariant* tensors:

$$u_\mu = g_{\mu\nu} u^\mu \qquad F_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} F^{\rho\sigma} \tag{25}$$

Thus the scalar product $u^\nu g_{\nu\mu} v^\mu$ can be simply written as $u_\mu u^\mu$.
We can transform the covariant tensors back to contravariant using the inverse of the metric tensor:

$$u^\mu = g^{\mu\nu} u_\nu \qquad F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \tag{26}$$

with

$$(g^{\cdot\cdot})(g_{\cdot\cdot}) = 1, \quad \text{also} \quad g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_{\ \gamma} \tag{27}$$

Here δ stands for the Kronecker-Delta symbol. Actually g is diaginal, and it is trivial to invert

$$g^{\cdot\cdot} = g_{\cdot\cdot} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{28}$$

For completeness, we also introduce $g_\mu^{\ \nu} = g^\mu_{\ \nu} = \delta_{\mu\nu}$, thus we can always insert a g: e.g. $F^{\mu\nu} = g^\mu_{\ \rho} F^{\rho\nu}$.

The derivative with respect to a space-time coordinate:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{29}$$

Homogeneous field equations in electrodynamics: $\partial_\nu \tilde{F}^{\mu\nu} = 0$

- $\mu = 0$: $\partial_\nu \tilde{F}^{0\nu} = -\vec{\nabla}\vec{B} = 0$
- $\mu = 1$: $\partial_\nu \tilde{F}^{1\nu} = \dot{B}_x + \left(-\frac{\partial E_y}{\partial z} + \frac{\partial E_z}{\partial y}\right) = \dot{B}_x + (\vec{\nabla} \times \vec{E})_x$
- $\mu = i$: $\partial_\nu \tilde{F}^{i\nu} = \left(\dot{\vec{B}} + \vec{\nabla} \times \vec{E}\right)_i = 0$

Inhomogeneous field equations: $\partial_\nu F^{\mu\nu} = -\mu_0 j^\mu$.

- $\mu = 0$: $\partial_\nu F^{0\nu} = -\vec{\nabla}\vec{E} = -\mu_0 \rho$
- $\mu = 1$: $\partial_\nu F^{1\nu} = \dot{\vec{E}}_x + \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}$
- $\mu = i$: $\partial_\nu F^{i\nu} = \left(\dot{\vec{E}} - \vec{\nabla} \times \vec{B}\right)_i = -j_i$

with $j^\mu = (\rho, \vec{j})$.

From $\partial_\nu F^{\mu\nu} = -\mu_0 j^\mu$ we obtain the continuity equation:

$$0 = -\frac{1}{\mu_0}\partial_\mu\partial_\nu F^{\mu\nu} = \partial_\mu j^\mu \tag{30}$$

or the same equation with non-relativistic notation ($j^\mu = (\rho, \vec{j})$):

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\vec{j} = 0 \tag{31}$$

This means that the electric change is a conserved quantity.
The equation of motion $\partial_\mu \tilde{F}^{\mu\nu} = 0$ is valid in the entire space time. So there must exist an A field, which we call *vector potential*.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{32}$$

It is easy to check that the homogeneous equation is always satisfied:

$$\partial_\alpha \tilde{F}^{\alpha\beta} = \partial_\alpha \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\left(\partial_\alpha\partial_\mu A_\nu - \partial_\alpha\partial_\nu A_\mu\right) = 0 \tag{33}$$

Using A one can build new scalars: $A_\mu A^\mu$, $\partial^\mu A_\mu$, or $A_\mu \partial^\mu \phi$.

Before we can actually solve the equations of motions in terms of the vector potential we observe new symmetry, that is unrelated to the Lorentz or Poincare groups.

Gauge invariance: Let's introduce ${A'}_\mu = A_\mu + \partial_\mu \Phi$ and use it in the field tensor.

$$
{F'}_{\mu\nu} = \partial_\mu \left(A_\nu + \partial_\nu \phi \right) - \partial_\nu \left(A_\mu + \partial_\mu \phi \right) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \tag{34}
$$

Gauge transformation: The replacement of the vector potential through ${A'}_\mu = A_\mu + \partial_\mu \Phi$
Gauge field: $A_\mu(x)$ the field that has this symmetry
Gauge fixing: We set up an equation for the vector potential. This is an extra equation in addition to the equation of motion. This equation shall be such that any gauge field can be transformed to obey it.
e.g. Lorentz gauge fixing $\partial_\mu A^\mu(x) \equiv 0$.
Any $A_\mu(x)$ can be transformed using a $\Phi(x)$ field to Lorentz gauge fixing where Φ is the solution of the equation:

$$
\partial_\mu \partial^\mu \Phi(x) = \partial_\mu A^\mu(x) \tag{35}
$$

After such a transformation $\partial_\mu F^{\mu\nu} = -j^\nu / \mu_0$ simplifies as

$$
\partial^\mu \left(\partial_\mu A^\nu - \partial^\nu A_\mu \right) = \partial^\mu \partial_\mu A^\nu(x) = -j^\nu / \mu_0 \tag{36}
$$

In the absence of matter the vector potential obeys a wave equation, it corresponds to the free propagation of light. There are other covariant wave equations, too, such as:

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \tag{37}$$

ϕ is a scalar field, each term in the equation of motion is a scalar (invariant under Lorentz transformaion)
This is the *Klein-Gordon equation*.
What physics does it correspond to?
Switching to Fourier space using the standard formulas, now in 4D:

$$\phi(x) = \int \frac{dk^4}{(2\pi)^4} e^{-ik_\mu x^\mu} \phi(k), \qquad \phi(k) = \int dx^4 e^{ik_\mu x^\mu} \phi(x) \tag{38}$$

with $k_\mu x^\mu = Et - \vec{p}\vec{x}$, we can write

$$-k_\mu k^\mu \phi(k) + m^2 \phi(k) = 0 \tag{39}$$

A solution is possible only if $k_\mu k^\mu = m^2$, this is the dispersion relation.
Quantum mechanics relates the wave number k to the momentum $p = k\hbar$.
Thus in $c = 1$ and $\hbar = 1$ units we end up with

$$E^2 - \vec{p}^2 = m^2, \tag{40}$$

which equation describes a relativistic particle of mass m.

The simple homogeneous Klein-Gordon equation can be extended with a source term:

$$(\partial^2 + m^2)\phi(x) = J(x) \tag{41}$$

This is still a covariant equation. Its solution is given by the theory of Green's functions. Suppose, we find a function Δ_R that satisfies:

$$(\partial^2 + m^2)\Delta_R(x-y) = -\delta^4(x-y) \tag{42}$$

$\Delta(x-y)$ is called the *propagator*. Then for any J:

$$\phi(x) = \phi_0(x) - \int d^4y \Delta(x-y)J(y) \tag{43}$$

($\phi_0(x)$ is any solution to the homogeneous eqation). To find Δ let us insert its Fourier transform into (42):

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)}\tilde{\Delta}(k) \tag{44}$$

so we get

$$(-k^\mu k_\mu + m^2)\tilde{\Delta}(k) = -1 \tag{45}$$

Let us first find Δ in real time, where the Fourier transform is introduced in space only:

$$\Delta(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\bar{\Delta}(\vec{k},t)e^{+i\vec{k}\vec{x}} \tag{46}$$

$$\bar{\Delta}(\vec{k},t) = \int d^3x\Delta(\vec{x},t)e^{-i\vec{k}\vec{x}} \tag{47}$$

Then transforming both sides of $(\partial^2+m^2)\Delta(x) = -\delta(x)$ we find (after integrating by parts)
$\int d^3xe^{-i\vec{k}\vec{x}}(\partial_\mu^2+m^2)\Delta(\vec{x},t) = \int d^3xe^{-i\vec{k}\vec{x}}(\partial_0^2+|\vec{k}|^2+m^2)\Delta(\vec{x},t) =$
$(\partial_0^2+\omega_k^2)\bar{\Delta}(\vec{k},t)$,
where we introduced $\omega_k^2 = m^2+|\vec{k}|^2$.
The other side: $\int d^3x(-\delta^4(\vec{x},t))e^{-i\vec{k}\vec{x}} = -\delta(t)$
So we actually have to solve for each $\vec{k}$

$$\ddot{\bar{\Delta}}(t)+\omega_k\bar{\Delta}(t) = -\delta(t) \tag{48}$$

One possible solution (*the retarded propagator*):

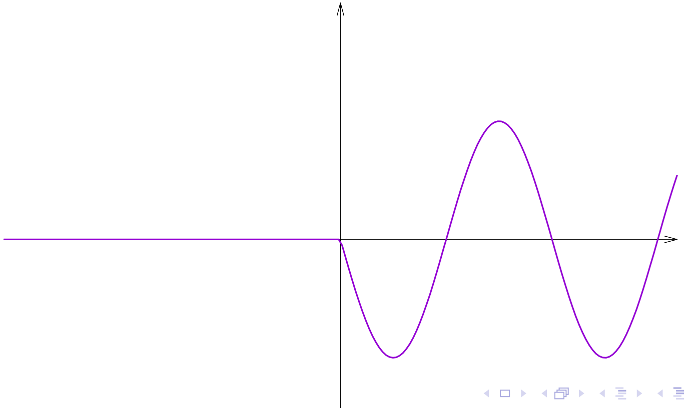
$$\bar{\Delta}_R(t) = \begin{cases} t\le 0 & 0 \\ t>0 & -\frac{\sin\omega_k t}{\omega_k} \end{cases} \tag{49}$$

Using this Green's function

$$\bar{\Delta}_R(t) = \begin{cases} t \leq 0 & 0 \\ t > 0 & -\frac{\sin \omega_k t}{\omega_k} \end{cases} \tag{50}$$

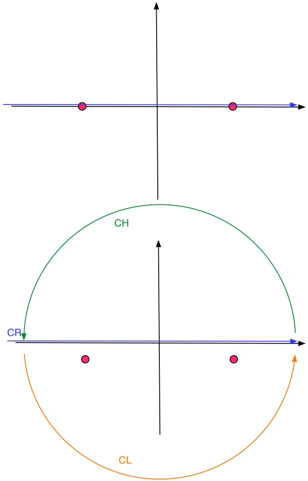
causality will be respected by the solution

$$\phi(x) = \phi_0(x) - \int d^4 y \Delta_R(x-y) J(y) \tag{51}$$



The same solution can also be found by making the Fourier transformation in time, too:

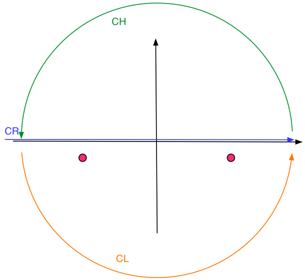
$$\bar{\Delta}(\vec{k}) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{(k_0 + \omega_k)(k_0 - \omega_k)} \tag{52}$$



The integral crosses two poles at $\pm\omega_k$, and is, thus ill-defined.

We shifted the poles to $\pm\omega_k - i\epsilon$.
We draw auxilliary contours CH and CL with an infinite radius.
We will use the Cauchy's integral theorem to calculate the integrals:

$$\int dk_0 \sum_j \frac{A_j}{k_0 - c_j} = \sum_j 2\pi i A_j \tag{53}$$



If $t < 0$, since $k_0 \to i\infty$ on the CH contour, we have $e^{-ik_0 t} \to e^{-\infty} = 0$. Then the integral on the CH contour is zero.

If $t > 0$, since $k_0 \to -i\infty$ we have $e^{-ik_0 t} \to e^{-\infty} = 0$. Then the integral on the CL contour is zero.

For $t < 0$

$$\int_{CR} = \int_{CR+CH} = 0, \text{there are no poles within the contour} \quad (54)$$

For $t > 0$

$$\int_{CR} = \int_{CR+CL} = \frac{1}{2\pi} e^{-i\omega_k t} \frac{(-2\pi i)}{2\omega_k} + \frac{1}{2\pi} e^{+i\omega_k t} \frac{(-2\pi i)}{-2\omega_k} \quad (55)$$

$$\bar{\Delta}(\vec{k}, t) = -\frac{\sin(\omega_k t)}{\omega_k} \quad (56)$$

which agrees with our previous result.

The poles can be shifted in four different ways. The other noteworthy option (Nr. 3 in this page) is the *time ordered* or *Feynman-propagator*.

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \tag{57}$$

1. *Causal retarded Green's function* G_R for poles at $k_0 = \pm\omega_k - i\epsilon$,

2. *Causal advanced Green's function* G_A for poles at $k_0 = \pm\omega_k + i\epsilon$,

3. *Time-ordered Green's function* G_F for poles at $k_0 = \pm(\omega_k - i\epsilon)$,

   This Green's function is the Feynman's propagator (1).

4. *Anti-time-ordered Green's function* G_{AT} for poles at $k_0 = \pm(\omega_k + i\epsilon)$,

We have seen in classical mechanics that the Newtonian equations of motion minimize a functional that we call *action*.

$$\delta S = 0\,. \tag{58}$$

The action is an integral of the Lagrange function over time:

$$S\left(a, b, \{q_i(t_a)\}, \{q_i(t_b)\}\right) = \int_a^b L\left(\{q_i\}, \{\dot{q}_i\}, t\right) dt \tag{59}$$

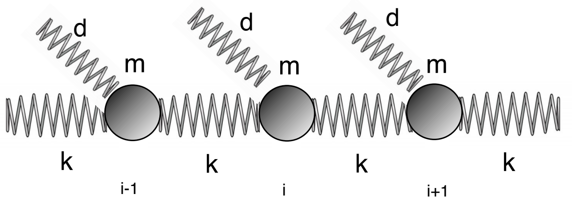
At the minimum of the action the Euler-Largange equations are fulfilled:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{60}$$

In most cases the Lagrange function is the difference between the kinetic and potential energies: $L = T - V$.
E.g. for a series of independent oscillators:

$$L\left(\{q_i\}, \{\dot{q}_i\}, t)\right) = \sum_i \frac{m_i}{2}\dot{q}_i^2 - \sum_i \frac{m_i\omega_i^2}{2} q_i^2 \tag{61}$$

Let's imagine an (infinite) chain of oscillators:



$$L\left(\{q_i\},\{\dot{q}_i\},t)\right) = \sum_i \left[\frac{m}{2}\dot{q}_i^2 - \frac{d_i}{2}q_i^2 - \frac{k}{2}(q_i - q_{i+1})^2\right] \qquad (62)$$

With $\phi_i = q_i(mk)^{1/4}$, $a = (m/k)^{1/2}$ and $\mu = (d/m)^{1/2}$ we can write L as

$$L\left(\{\phi_i\},\{\dot{\phi}_i\},t\right) = \sum_i \left[\frac{a}{2}\dot{\phi}_i^2 - \frac{a\mu^2}{2}\phi_i^2 - \frac{1}{2a}(\phi_i - \phi_{i+1})^2\right] \qquad (63)$$

Then the Euler-Lagrange equations (divided by a):

$$\ddot{\phi}_i + \mu^2\phi_i + \frac{1}{a^2}(2\phi_i - \phi_{i+1} - \phi_{i-1}) = 0 \qquad (64)$$

Taking the $a \to 0$ limit and generalizing to 3+1 dimensions:

$$
\begin{aligned}
L\left(\{\phi_i\},\{\dot{\phi}_i\},t\right) &= \sum_i \left[\frac{a}{2}\dot{\phi}_i^2 - \frac{a}{2}\mu^2\phi_i^2 - \frac{1}{2a}(\phi_i - \phi_{i+1})^2\right] \\
\to \quad & \int dx \left[\frac{1}{2}(\dot{\phi}(x))^2 - \frac{1}{2}\mu^2(\phi(x))^2 - \frac{1}{2}(\vec{\nabla}\phi(x))^2\right] \\
\to \quad & L = \int dx [\frac{1}{2}\partial_\nu\phi(x)\partial^\nu\phi(x) - \frac{1}{2}\mu^2\phi(x)^2]
\end{aligned}
\tag{65}
$$

We call the quantity under $\int dx$ the *Lagrange density*.

$$
\mathcal{L} = \frac{1}{2}\partial_\nu\phi(x)\partial^\nu\phi(x) - \frac{1}{2}\mu^2\phi(x)^2 \tag{66}
$$

Then we get for the equation of motion:

$$
\begin{aligned}
\ddot{\phi}_i + \nu^2\phi_i + \frac{1}{a^2}(2\phi_i - \phi_{i+1} - \phi_{i-1}) &= 0 \\
\to \quad \ddot{\phi}(x) - \triangle\phi(x) + \mu^2\phi_i &= 0 \\
\to \quad (\partial_\nu\partial^\nu + \mu^2)\phi(x) &= 0\,, \quad \textit{The Klein-Gordon equation}
\end{aligned}
\tag{67}
$$

With a limit process we get

$$\mathcal{L} = \frac{1}{2}\partial_\nu\phi(x)\partial^\nu\phi(x) - \frac{1}{2}\mu^2\phi(x)^2 \quad \rightarrow \quad (\partial_\nu\partial^\nu + \mu^2)\phi(x) = 0 \tag{68}$$

We can generalize the rule for getting the equation of motion:

Lagrange function $\rightarrow$ *Lagrange functional*

$$\begin{aligned}
0 \equiv \frac{\delta S}{\delta\phi(x)} &= \frac{\int \mathcal{L}(\phi, \partial\phi, y) d^4y}{\delta\phi(x)} \\
&= \int \left[\frac{\partial\mathcal{L}}{\partial\Phi(y)} \underbrace{\frac{\delta\Phi(y)}{\delta\Phi(x)}}_{\delta(x-y)} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi(y))} \underbrace{\frac{\delta(\partial_\mu\Phi(y))}{\delta\Phi(x)}}_{\partial_\mu\delta(y-x)} \right] d^4y \\
&= \int \left[\frac{\partial\mathcal{L}}{\partial\Phi(y)}\delta(x-y) - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi(y))}\delta(y-x) \right] d^4y \\
&= \frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} \right)
\end{aligned} \tag{69}$$

The Lorentz tranformation (the Λ matrices) form a algebraic group:

$$\Lambda(v_1)\Lambda(v_2) = \Lambda(v_3) \tag{70}$$

$$\Lambda(v_1)\Lambda(-v_1) = \Lambda(0) = 1 \tag{71}$$

with $v_3 = \frac{v_1+v_2}{1+v_1 v_2}$. Not only boosts, but rotations are also part of the Lorentz group, they are described by the Euler angles.
These transformations can be extended by including the translations:
$x^\mu \to x^\mu + a^\mu$.
Lorentz group: Reflections, rotations and Lorentz boost
Poincaré group: Reflections, rotations, Lorentz boost and translations
The Lorentz group is described by 6 parameters (has 6 generators).
The Poincaré group has 10 parameters in total.
If the translation is a symmetry, then the conserved quantities related to these symmetries are called energy and momentum.

Sometimes a group is defined though a certain type of matrices (e.g. 2×2 unitary matrices: $U(2)$). But the elements of the algebra can often be mapped to other (larger) matrices that obey the same rules. There can also be more abstract groups that were originally not defined through matrices, but later we find a set of matrices that fulfil the same algebraic relations.
For a group G we call the mapping D: $G \to K^{2n}$ an n dimensional *representation* of the group if

$$\begin{aligned} & \forall g \in G \quad \exists D(g) \in K^{2n} \; (n \times n \text{ matrix}) \\ \forall a, b \in G \quad : \quad & D(a)D(b) = D(ab) \; (\text{matrix product}) \end{aligned} \tag{72}$$

Some groups are defined as a set of matrices: they are given in terms of their defining representation.

Let's have e.g. the 1x1 unitary matrices, the U(1) group. The multiplication rules are the same as for the normal complex numbers. Each number has a magnitude of 1. The product of such numbers will also have a magnitude of 1. The algebra reduces to addition of phases with a period of 2π.

The 2x2 unitary matries with a fixed determinant $=1$ form the group SU(2). It has 3 real parameters, but can be represented by e.g. 3×3 matries, too.

There are finite groups e.g. the rotations of a Rubik's cube: a finite set of discrete transformations.
Lie groups, on the other hand, are formed by continuous transformations. They are described by a finite number of parameters, and one can expand in these (ϕ_j) parameters:

$$D(\phi_j) = \exp\left(i\sum_j \phi_j X_j\right) = D(0) + i\sum_j \phi_j X_j = 1 + i\sum_j \phi_j X_j \tag{73}$$

Statement: The X matrices are Hermitian.
The X_j matrices are the *generators* of the Lie group in the given representation.
Example SO(2): 2×2 matrices with det=1, they describe the 2D rotations with a single parameter ϕ

$$a(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \tag{74}$$

In this case $X = -i\frac{\partial a(\phi)}{\partial \phi} = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2$

Previously we used the Lorentz (Poincare) group elements to transform coordinates. How do fields transform.
Let's first consider a *scalar field*: its value at the same physical location is independent on the reference frame.

$$\begin{aligned}\delta f(x) &= f'(x') - f(x) \\ &= f'(x+\delta x) - f(x) \\ &= f'(x) - f(x) + \delta x^\mu \partial_\mu f' + \mathcal{O}(\delta x^2) \\ &= f'(x) - f(x) + \delta x^\mu \partial_\mu f + \mathcal{O}(\delta x^2) \stackrel{!}{=} 0 \end{aligned} \tag{75}$$

The *generator* of an infinitesmial translation:

$$-i\frac{\delta[f'(x) - f(x)]}{\delta x^\mu} = i\partial_\mu f =: -P_\mu f \tag{76}$$

P is a linear operator. We call the generator of the translation: *momentum*, and the genertor of the time translation: *energy*. Together: *four-momentum*.

Let us now do an infinitesimal Lorentz transformation on the coordinates:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{77}$$

with $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \alpha^{\mu}{}_{\nu}$.
Now the equation $\Lambda^T g \Lambda = g$ puts a constrain on α: $g_{\nu\rho}\alpha^{\rho}{}_{\mu} + g_{\mu\rho}\alpha^{\rho}{}_{\nu}$. From this we have

$$\alpha_{\mu\nu} = -\alpha_{\nu\mu} \tag{78}$$

A 4×4 anti-symmetric matrix has 6 independent parameters:

$$f'(x) - f(x) = -\alpha^{\mu}{}_{\rho} x^{\rho} \partial_{\mu} f = -i\frac{1}{2}\alpha^{\rho\mu} L_{\rho\mu} f \tag{79}$$

with

$$L_{\rho\sigma} = i\left(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}\right) \tag{80}$$

L is the generator for the generalized rotations in space time.
In 3D space we have rotations only, there $L_{jk} = i(x_j \nabla_k - x_k \nabla_j)$

Previous slide: $-i\partial_{\mu} = P_{\mu}$, thus we can write

$$L_{\rho\sigma} = x_{\sigma} P_{\rho} - x_{\rho} P_{\sigma} \tag{81}$$

We recognize the form of the angular momentum operator.

Let's study the components of this generator matrix:

$$L_{\rho\sigma} = \begin{pmatrix} 0 & K_x & K_y & K_z \\ -K_x & 0 & J_z & -J_y \\ -K_y & -J_z & 0 & J_x \\ -K_z & J_y & -J_x & 0 \end{pmatrix} \tag{82}$$

$\vec{J}$ stands for the ordinary angular momentum, which is a pseudo vector. $\vec{K}$ is a vector, it describe the boosts. The formula can be turned around:

$$J_i = \frac{1}{2}\epsilon_{ijk} L_{jk} \tag{83}$$

$$K_i = L_{0i} \tag{84}$$

Using the anti-symmetry of L we can further write

$$J_i = \frac{1}{2}\epsilon_{ijk}(x_k P_j - x_j P_k) \tag{85}$$

If we interpret the four vectors as $x_\mu = (x_0, -\vec{x})$ but $P_\mu = (P_0, \vec{P})$ then we arrive at the familiar

$$\vec{J} = \vec{x} \times \vec{P} \tag{86}$$

$\vec{P}$, $\vec{J}$ and $\vec{K}$ are generators, i.e they are linear operators operating on the fields.

We have not introdued any quantum effects so far, yet we can already deduce some commutation relations. These simply follow from the Lie Algebra of the Poincare group. For example:

$$[x^i, P_j] = -i[x^i, \partial_j] = i\delta_{ij} \tag{87}$$

Now if we use this for the angular momentum $J_i = \epsilon_{ijk} x^j P_k$

$$\begin{aligned}
[J_a, J_b] &= \epsilon_{ajk}\epsilon_{bmn}[x^j P_k, x^m P_n] = \epsilon_{ajk}\epsilon_{bmn}(-ix^j \delta_{km} P_n + ix^m \delta_{nj} P_k) \\
&= -i\epsilon_{ajk}\epsilon_{bkn} x^j P_n + i\epsilon_{ajk}\epsilon_{bmj} x^m P_k \\
&= -i\epsilon_{ajk}\epsilon_{bkn} x^j P_n + i\epsilon_{akn}\epsilon_{bjk} x^j P_n = -i(\epsilon_{ajk}\epsilon_{nbk} - \epsilon_{nak}\epsilon_{bjk}) P_j x_n \\
&= -i(\delta_{an}\delta_{jb} - \delta_{ab}\delta_{jn} - \delta_{nb}\delta_{aj} + \delta_{ab}\delta_{nj}) x^j P_n \\
&= -i\epsilon_{abc}\epsilon_{cnj} x^j P_n = i\epsilon_{abc}\epsilon_{cjn} x^j P_n = i\epsilon_{abc} J_c
\end{aligned} \tag{88}$$

$$[J_a, J_b] = i\epsilon_{abc} J_c \tag{89}$$

Statement: The commuatiton relations are independent on the representation. (Representation: on what sort of fields or object the generators are operating). Algebras are classified by their communation relations. Here we reconginze algebra of the group of 3D rotations.

Attention: The su(2) algebra has the same commutation relation.

In three dimensions a generic rotation has three parameters:

$$R(\psi,\theta,\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can get the generators as: $-i\partial R/\partial\varphi_i|_{\varphi\equiv 0}$.

$$T_1 = -i\frac{\partial R}{\partial\phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad (90)$$

$$T_2 = -i\frac{\partial R}{\partial\theta} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad (91)$$

$$T_3 = -i\frac{\partial R}{\partial\psi} = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (92)$$

So for infinitesimal transformations: $R \approx 1 + iT_1\phi + iT_2\theta + iT_3\psi$.
It is easy to check that the generators fulfil the commutation relation:

$$[T_i, T_j] = i\epsilon_{ijk}T_k \quad (93)$$

In the case of a scalar field an infinitesimal Lorentz transformation ($\alpha^{\mu\nu} \ll 1$) is written as:

$$\Phi'(x') - \Phi(x) = -\frac{1}{2}\alpha^{\rho\sigma} i(x_\rho \partial_\sigma - x_\sigma \partial_\rho)\Phi(x) \tag{94}$$

A *vector* field has the defining feature that its components transform like a vector (e.g. A_μ). One can construct such a vector field from a scalar field as $\partial_\mu \phi$. We work out how $\partial_\mu \phi$ transforms and learn how to take care of the indices.

$$(\partial_\mu \Phi)'(x') - (\partial_\mu \Phi)(x) = \frac{1}{2}\alpha^{\rho\sigma}(x_\rho \partial_\sigma - x_\sigma \partial_\rho)\partial_\mu \phi - \frac{i}{2}\alpha^{\rho\sigma}(S_{\rho\sigma})_\mu{}^\nu \partial_\nu \phi \tag{95}$$

Now S is a Tensor with four indices:

$$(S_{\rho\sigma})_\mu{}^\nu = i\left(g_{\rho\mu} g^{\nu\sigma} - g_{\sigma\mu} g^\nu{}_\rho\right) \tag{96}$$

Thus $L_{\rho\sigma} = i(x_\rho \partial_\sigma - x_\sigma \partial_\rho)$ and $S_{\rho\sigma}$ together generate the rotation. Thus for a generic vector field:

$$A'_\mu(x') - A_\mu(x) = -\frac{i}{2}\alpha^{\rho\sigma}(M_{\rho\sigma})_\mu{}^\nu A_\nu(x) \tag{97}$$

with $(M_{\rho\sigma})_\mu{}^\nu = L_{\rho\sigma} g_\mu{}^\nu + (S_{\rho\sigma})_\mu{}^\nu$.
L rotates die inhomogeneous structure of the field, S rotates the indices, S stands for ***Spin****.*

Our starting point now is the commutation relation

$$[J_a, J_b] = i\epsilon_{abc} J_c \tag{98}$$

From this simple algebraic rule many statements follow.
The combination

$$J^2 = \sum_a J_a J_a \tag{99}$$

commutes with all other elements of the algebra

$$[J^2, J_a] = 0 \tag{100}$$

and as such is called a *Casimir operator*.
But then in any representation J^2 will be proportional to the identity matrix.

$$J^2 = j(j+1) \tag{101}$$

It has been shown (see Quantum Mechanics lecture) that j is either an integer or half intereg, depending on the representation.

E.g for 3D rotations of vectors (3×3 matrices) we have $j = 1$.
Scalars do not have to be rotated at all, then $j = 0$.
The choice $j = 1/2$ introduces the fundamental fermion fields.

At the same time one can show that J_a has eigenvalues ranging from $m = -j \ldots j$. where the representation is $2j + 1$ dimensions.

How comes that the angular momentum is quantized without talking about quantum mechinics?
The commutiation relations simply follow from the algebraic rules of rotations.

$$[J_a, J_b] = i\epsilon_{abc}J_c \tag{102}$$

Let's set the so far neglected dimensions:

$$[J] = [\mathrm{angularmomentum}] = [\mathrm{momentum}\ \times\ \mathrm{lenguth}] = \mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}}\mathrm{m} = \mathrm{Js} \tag{103}$$

We wrote the commutation relation ($\hbar = 1, c = 1$) in natural units. Otherwise it reads:

$$[J_a, J_b] = i\epsilon_{abc}\hbar J_c \tag{104}$$

and

$$J = \sqrt{j(j+1)}\hbar, \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2} \dots \tag{105}$$

The half indices are always linked to internal degrees of freedom, for ordinary rotations of a scalar or vector we have only integer $j = 0, 1, 2 \dots$.

Suppose, we find two representations for a group G:

$$\begin{aligned} D_1: \quad & G \to K^{2n_1} \quad \forall g, h \in G: D_1(g)D_1(h) = D_1(gh) \\ D_2: \quad & G \to K^{2n_2} \quad \forall g, h \in G: D_2(g)D_2(h) = D_2(gh) \end{aligned} \tag{106}$$

(this means that $D_1(g)$ are $n_1 \times n_1$ and $D_2(g)$ are $n_2 \times n_2$ matrices)
A *direct sum* of the both representations $D_3 = D_1 \oplus D_2$ is a new representation:

$$D_3 : G \to K^{2(n_1+n_2)} \quad \forall g \in G : D_3(g) = \left(\begin{array}{c|c} D_1(g) & 0 \\ \hline 0 & D_2(g) \end{array}\right) \tag{107}$$

All D_3 matrices are block diagonal.
If a representation is block diagonal, or it can be rotated such that all representation matrices become block diagonal by the same unitary rotation matrix, then that representation is called *reducible*.

$$\exists U, D_1, D_2 : \forall g \in G : UD(g)U^{-1} = \left(\begin{array}{c|c} D_1(g) & 0 \\ \hline 0 & D_2(g) \end{array}\right) \tag{108}$$

If no such rotation exists, the representation is *irreducible*.

Now for all $n \in \mathbf{Z}^+$ there is a set of $n \times n$ matrices that fulfil the equation

$$[J_a, J_b] = i\epsilon_{abc}\hbar J_c \tag{109}$$

Space	$J^2 = \sum_a J_a J_a$	eigenvalues of the J_a matrices
1×1	0	0
2×2	$\frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$-\frac{1}{2}, +\frac{1}{2}$
3×3	$2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$-1, 0, +1$
4×4	$\frac{15}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$-\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$
$(2j+1) \times (2j+1)$	$j(j+1)\mathbf{1}$	$-j, -j+1, \ldots, j-1, j$

All these belong to a separate irreducible representations of the algebra.

Proof: Quantum Mechanics lecture

Let us now come back to the Lorentz group. For a generic field with some indices:

$$\Phi'_i(x') - \Phi_i(x) = -\frac{i}{2}\alpha^{\rho\sigma}\left(L_{\rho\sigma}\delta_{ij} + (S_{\rho\sigma})_{ij}\right)\Phi_j(x) = -\frac{i}{2}\alpha^{\rho\sigma}(M_{\rho\sigma})_{ij}\Phi_j(x) \quad (110)$$

$$J_a = \frac{1}{2}\epsilon_{abc}M_{bc} \quad (111)$$

$$K_a = M_{0a} \quad (112)$$

The algebraic rules for J and K are well known.
(For a scalar field we had $M_{\rho\sigma} = L_{\rho\sigma} = i(x_\rho\partial_\sigma - x_\sigma\partial_\rho)$, and the commutators are easily obtained. Since the commutation relations are independent of the representation, we can generalize them for more generic fields.)

$$[J_a, J_b] = i\epsilon_{abc}J_c \quad (113)$$

$$[K_a, K_b] = -i\epsilon_{abc}J_c \quad (114)$$

$$[J_a, K_b] = i\epsilon_{abc}K_c \quad (115)$$

Only the first line (rotation) can be interpreted as an $\mathfrak{su}(2)$ generator algebra.

$$
\begin{aligned}
[J_a, J_b] &= i\epsilon_{abc} J_c &(116)\\
[K_a, K_b] &= -i\epsilon_{abc} J_c &(117)\\
[J_a, K_b] &= i\epsilon_{abc} K_c &(118)
\end{aligned}
$$

However, one can introduce a linear combination

$$
N_a \equiv \frac{1}{2}\left(J_a + iK_a\right), \quad M_a \equiv \frac{1}{2}\left(J_a - iK_a\right) \tag{119}
$$

where two decoupled $\mathfrak{su}(2)$ algebras emerge.

$$
\begin{aligned}
[N_a, M_b] &= 0 &(120)\\
[N_a, N_b] &= i\epsilon_{abc} N_c &(121)\\
[M_a, M_b] &= i\epsilon_{abc} M_c &(122)
\end{aligned}
$$

J and K are actually unitary representations, this makes J and K Hermitian.
Thus $N_a^{+} = M_a$.
But then there are two independent Casimir operators:

$$
\sum_a N_a N_a = n(n+1) \qquad \sum_b M_b M_b = m(m+1) \tag{123}
$$

and the representation is chosen independently. We have two quantum numbers n and m.

In the representations of the two $\mathfrak{su}(2)$ algebras n and m can be $0, \frac{1}{2}, 1, \ldots$ as usual.
This gives us several possible mathematical possibilities for a field on which the Lorentz transformation operates.

- $(0, 0)$: Scalar field (a single field component)
- $\left(\frac{1}{2}, 0\right)$: Spinor field (one index), $\Psi_{L\,i}(x)$, $i = 1, 2$
- $\left(0, \frac{1}{2}\right)$: A different spinor field (one index), $\Psi_{R\,i}(x)$, $i = 1, 2$
- $\left(\frac{1}{2}, \frac{1}{2}\right)$: Two indices $\Phi_{ij}(x)$, $i, j = 1, 2$
  In combination there are four field components. One can use this to pack a four vector into this structure:

$$A^\mu = \begin{pmatrix} A^0 + A^3 & A^1 - iA^2 \\ A^1 + iA^2 & A^0 - A^3 \end{pmatrix} = A^0 + \sum_a A^a \sigma^a \qquad (124)$$

σ^a, with $a = 1, \ldots, 3$ are the Pauli matrices.

The Lorentz group can also be represented on fields with one internal index: $\Phi_i(x)$, $i = 1, 2$.

$$\Phi'_i(x') - \Phi_i(x) = -\frac{i}{2}\alpha^{\rho\sigma}\left(L_{\rho\sigma}\delta_{ij} + (S_{\rho\sigma})_{ij}\right)\Phi_j(x) \tag{125}$$

The derivatives are in L. S will do the transformation in the internal space. For any field S has two four-indices. In addition, S has now two internal indices (spinor index).
First, let us focus on the rotation group. For that only the spatial indices will matter.

$$(S_a)_{ij} = \frac{1}{2}\epsilon_{abc}(S_{bc})_{ij}, \qquad i,j = 1,2 \quad a,b,c = 1,2,3 \tag{126}$$

If an electron is at stillstand *(this argument fails in the case of a massless particle)*, then S alone is responsible for the representation of the rotation group (and its algebra):

$$[S_a, S_b] = i\epsilon_{abc}S_c \tag{127}$$

So we must find 2×2 Hermitian matrices that fulfil Eq. (127).

The *Pauli matrices* are the following three 2×2 matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{128}$$

Their Product fulfils the algebra:

$$\sigma_a \sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c \tag{129}$$

Then, taking $S_a = \frac{1}{2}\sigma_a$, we found a representation of

$$[S_a, S_b] = i\epsilon_{abc} S_c \tag{130}$$

in terms of 2×2 matrices.
The Pauli matrices are Hermitian, eigenvalues are -1 and +1.
The determinants: $\det \sigma_a = -1$ and $\mathrm{Tr}\, \sigma_a = 0$, $\forall a = 1 \ldots 3$
Since $\sigma_i \sigma_i = 1$ for $i = 1, 2, 3$. The sum is a diagonal matrix:

$$\sum_a S_a S_a = \frac{3}{4}\,. \tag{131}$$

This means that this sum commutes with all S_a matrices (generators). Such operators (matrices) we call a *Casimir operator*.

Identities with Pauli matrices.

- $\left[\frac{1}{2}\sigma_a, \frac{1}{2}\sigma_b\right] = i\epsilon_{abc}\frac{1}{2}\sigma_c$
- $\left\{\frac{1}{2}\sigma_a, \frac{1}{2}\sigma_b\right\} = \delta_{ab}\frac{1}{2}$
- $\sigma_2\sigma_a\sigma_2 = -\sigma_a^*$
- $\mathrm{Tr}\,\sigma_a = 0$
- $\mathrm{Tr}\,\sigma_a\sigma_b = 2\delta_{ab}$
- $\mathrm{Tr}\,\sigma_a\sigma_b\sigma_c = 2i\epsilon_{abc}$
- $\mathrm{Tr}\,\sigma_a\sigma_b\sigma_c\sigma_d = 2(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{cb})$

The Pauli matrices are quite often written as vectors: $\vec{\sigma}$, e.g.:

$$\vec{v}\vec{\sigma} = \sum_{a=1}^{3} v_a\sigma_a \tag{132}$$

is a complex 2×2 matrix. It is easy to check that

$$(\vec{a}\vec{\sigma})(\vec{b}\vec{\sigma}) = (\vec{a}\vec{b}) + i(\vec{a} \times \vec{b}) \times \vec{\sigma} \tag{133}$$

$$(\vec{v}\vec{\sigma})^{2n} = |\vec{v}|^{2n}, \qquad (\vec{v}\vec{\sigma})^{2n+1} = |\vec{v}|^{2n}\vec{v}\vec{\sigma}, \quad n \in \mathcal{N} \tag{134}$$

Spinors have a peculiar behaviour when rotated:
In a 3 dimensional representation ($j = 1$) the rotation around the z axis is written in this matrix form:

$$\vec{v}' = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & & 1 \end{pmatrix} \vec{v} = \exp\left\{ i\psi \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = e^{i\psi T_3} \vec{v} \tag{135}$$

The eigenvalues of the generators T_i are $-1, 0, +1$ for all $i = 1, 2, 3$. So for a full 2π rotation

$$\psi = 2\pi \quad \longrightarrow \quad e^{i\psi T_3} = \mathbf{1} \tag{136}$$

In the $j = 1/2$ representation, however the eigenvalues of $S_i = \sigma_i/2$ are $\pm\frac{1}{2}$. Then the same rotation results in

$$\Phi' = e^{i\psi S_3}\Phi = \left[e^{+i\frac{1}{2}\psi} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + e^{-i\frac{1}{2}\psi} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \Psi \tag{137}$$

$$\Phi' = \begin{pmatrix} e^{+i\frac{1}{2}\psi} & 0 \\ 0 & e^{-i\frac{1}{2}\psi} \end{pmatrix} \Phi \tag{138}$$

So 2π rotation results a sign change. Thus the sign of an electron field must not have any physical meaning.

So far we talked about the rotation group and the corresponding Lie algebra.
But from which group do the $S_a = \sigma_a/2$ generators be derived?
The matrices

$$U(\vec{\alpha}) = e^{i\vec{\alpha}\frac{1}{2}\vec{\sigma}} \tag{139}$$

are unitary, 2×2 complex matrices ($\alpha_i \in \mathbf{R}$).

$$U(\vec{\alpha})^{-1} = U(-\vec{\alpha}) = e^{-i\vec{\alpha}\frac{1}{2}\vec{\sigma}} = U^+(\vec{\alpha}) \tag{140}$$

(in the last step we used that $\sigma_a^+ = \sigma_a$).
The determinant can be calculated as the product of the eigenvalues. (The matrix $\vec{v}\vec{\sigma}$ are $\pm|\vec{v}|$.)

$$\det U(\vec{\alpha}) = e^{\frac{i}{2}|\vec{\alpha}|}e^{-\frac{i}{2}|\vec{\alpha}|} = 1 \tag{141}$$

The unitary 2×2 matrices with $\det = 1$ form the SU(2) group.
Its generators are the $S_a = \frac{1}{2}\sigma_a$, $a = 1, 2, 3$ matrices.
Question: are there other generators of SU(2)? How many parameters are there in SU(2)?

Let us construct field equations that respect the rotation symmetry. (First, we do in in the non-relativistic quantum mechanics):

$$E = \frac{mv^2}{2} + V(x) = \frac{p^2}{2m} + V(x) \tag{142}$$

Let E, p and x be represented in the vector space of some ψ fields.

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}(\vec{\nabla})^2\psi + V(x)\psi \tag{143}$$

If ψ is in the scalar representation (has no index), then we get back Schrödinger's equation.i
The angular momentum will become apparent as soon as we introduce e.g. a homogeneous magnetic field, with an interaction energy of $E_m = -\vec{\mu}\vec{B}$, with the magnetic moment $\vec{\mu} = \frac{q}{2m}\vec{L}$.

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2m}(-i\vec{\nabla})^2\psi + V(x)\psi - \frac{q}{2m}\vec{L}\vec{B}\psi \tag{144}$$

The full angular monetum of a particle with spin $j = 1/2$ is not $\vec{L}$, but $\vec{L}$ and $\vec{S}$ together.
Taking the spin inte account with $\vec{S} = \vec{\sigma}/2$ we get the *Pauli equation*:

$$i\frac{\partial \psi_i}{\partial t} = \frac{1}{2m}(-i\vec{\nabla})^2\psi_j + V(x)\psi - \frac{q}{2m}(\vec{L} + g\vec{S}_{ij})\vec{B}\psi_j \tag{145}$$

Here a new constant appears: $g - factor$, the dimensionless magnetic moment: the spin couples to the magnetic field not exacty as the rest of the angular momentum.

positron	$g = 2.0023193048(8)$
proton	$g = 5.585694702(17)$
neutron	$g = -3.82608545(90)$
Dirac equation	$g = 2$

For spin-1/2 particles large deviations from $g = \pm 2$ are hints for internal structure. Small deviations are caused by radiative corrections.
$\vec{B}$ is a pseudo vector. The full equation has a spinor value. So $\vec{B}$ must couple to an other pseudo vector ($\vec{L}$ or $\vec{S}$) to produce a scalar of the rotation group (no four-index).
S_i commutes with L_j and both L^2 and S^2 are Casimir operators, but only the sum ($\vec{J} = \vec{L} + \vec{S}$) is conserved.

There are two types of spinor representations. We will name them L and R. Since these are non-identical representations the Lorentz transformation will be realized by different matrices:

$$\psi_L(x) \to \psi'_L(x') = \Lambda_L(x)\psi_L(x) \quad \text{for} \quad \left(\frac{1}{2}, 0\right) \tag{146}$$

$$\psi_R(x) \to \psi'_R(x') = \Lambda_R(x)\psi_R(x) \quad \text{for} \quad \left(0, \frac{1}{2}\right) \tag{147}$$

$\Lambda_{R,L}$ are 2×2 complex matrices, they depend on the rotation angle $\vec{\omega}$ and the boost vector $\vec{\nu}$.

$$\vec{J}_{L/R} = \frac{1}{2}\vec{\sigma}, \quad \vec{K}_{L/R} = \mp\frac{i}{2}\vec{\sigma} \tag{148}$$

Note that K is not Hermitian, but it obeys the commutation relations. For simultaneous rotation and boost:

$$\Lambda_L(\vec{\omega}, \vec{\nu}) = e^{\frac{i}{2}\vec{\sigma}(\vec{\omega} - i\vec{\nu})}, \quad \Lambda_R(\vec{\omega}, \vec{\nu}) = e^{\frac{i}{2}\vec{\sigma}(\vec{\omega} + i\vec{\nu})} \tag{149}$$

The transformation matrices

$$\Lambda_L(\vec{\omega},\vec{\nu}) = e^{\frac{i}{2}\vec{\sigma}(\vec{\omega}-i\vec{\nu})}\,, \quad \Lambda_R(\vec{\omega},\vec{\nu}) = e^{\frac{i}{2}\vec{\sigma}(\vec{\omega}+i\vec{\nu})} \tag{150}$$

feature lot's of identities:

- $\Lambda_L(\vec{\omega},\vec{\nu}) = \Lambda_R(\vec{\omega},-\vec{\nu})$
- $\Lambda^{-1}_{L/R}(\vec{\omega},\vec{\nu}) = \Lambda_{L/R}(-\vec{\omega},-\vec{\nu})$ but $\Lambda^{+}_{L/R}(\vec{\omega},\vec{\nu}) = \Lambda_{L/R}(-\vec{\omega},\vec{\nu})$
- $\Lambda_L^{-1} = \Lambda_R^{+}$ und $\Lambda_R^{-1} = \Lambda_L^{+}$, thus $\Lambda_R^{*} = (\Lambda_L^{-1})^T = (\Lambda_L^T)^{-1}$
- $\sigma^2\Lambda_L\sigma^2 = \sigma^2 e^{\frac{i}{2}\vec{\sigma}(\vec{\omega}-i\vec{\nu})}\sigma^2 = e^{-\frac{i}{2}\vec{\sigma}^*(\vec{\omega}-i\vec{\nu})} = \left[e^{\frac{i}{2}\vec{\sigma}(\vec{\omega}+i\vec{\nu})}\right]^* = \Lambda_R^*$
- $\Lambda_L^T\sigma^2\Lambda_L = \sigma^2$ and $\Lambda_R^T\sigma^2\Lambda_R = \sigma^2$

Consequently for an L-spinor $\sigma^2\psi_L^*$ is in the R representation

$$\sigma^2\psi_L^* \to \sigma^2\Lambda_L^*\psi_L^* = \sigma^2\Lambda_L^*\sigma^2\sigma^2\psi_L^* = \Lambda_R\sigma^2\psi_L^* \tag{151}$$

but also: $\sigma^2\psi_R^*$ is in the L representation.
and, for two L spinors: $\chi_L^T\sigma^2\psi_L$ is a scalar.

$$\chi_L^T\sigma^2\psi_L \to \psi_L^T\Lambda_L^T\sigma^2\Lambda_L\psi_L = \chi_L^T\sigma^2\psi_L \tag{152}$$

From the previous slide we learned that $\sigma^2\psi_R^*$ is an L spinor and $\chi_L^T\sigma^2\psi_L$ is as scalar. Combining the two:

$$i(\sigma^2\psi_R^*)^T\sigma^2\psi_L = i\psi_R^+\psi_L \tag{153}$$

is a Lorentz scalar. However, $\psi_L^+\psi_L$ is not a scalar:

$$\psi_L^+\psi_L \to \psi_L^+ e^{\vec{\sigma}\vec{\nu}}\psi_L = \psi_L^+\psi_L + \vec{\nu}\psi_L^+\vec{\sigma}\psi_L + \mathcal{O}(\nu^2) \tag{154}$$

The rotation is represented with unitary matrices, and for pure rotations $(\vec{\nu}=0)$ $\psi_L^+\psi_L$ is invariant. Let's see how the new term transforms.

$$\psi_L^+\sigma^a\psi_L \to \psi_L^+ e^{\frac{1}{2}\vec{\sigma}\vec{\nu}}\sigma^a e^{\frac{1}{2}\vec{\sigma}\vec{\nu}}\psi_L = \psi_L^+\sigma^a\psi_L + \nu^a\psi_L^+\psi_L + \mathcal{O}(\nu^2) \tag{155}$$

Eqs. (154) and (155) together are the usual transformation rules for four vectors (V):

$$\delta V^\mu = \alpha^{\mu\nu}g_{\nu\rho}V^\rho\,, \qquad -\alpha^{i0} = \alpha^{0i} = -\nu^i \tag{156}$$

If we introduce the identity matrix as a fourth Pauli matrix

$$\sigma^\mu = \left(\mathbf{1},\sigma^1,\sigma^2,\sigma^3\right) \tag{157}$$

than we can write the transformations in the four vector notation.

Our aim is to write down a possible equation of motion for the spinor fields Ψ_L und Ψ_R.
To introduce time evolution we must use ∂_μ, its index will pair with σ^μ.
We know that

$$\Psi_L^+ \sigma^\mu \Psi_L : \quad \text{vector} \;\Rightarrow\; \Psi_L^+ \sigma^\mu \partial_\mu \Psi_L : \quad \text{scalar} \tag{158}$$

If this is a scalar, we could use it as a Lagrangian of a new theory. From this Lagrangian the Euler-Lagrange equation will look like:

$$i\sigma^\mu \partial_\mu \Psi_L = 0 \tag{159}$$

This already describes something: chiral (massless) fermions, or the free neutrinos of the standard model.
Ψ_L *has two components, but this are not the two spin components from the Pauli equation.*
Analogously, for the spinor in the R representation we have

$$i\bar{\sigma}^\mu \partial_\mu \Psi_R = 0 \tag{160}$$

where we introduced $\bar{\sigma}^\mu = (\mathbf{1}, -\sigma^1, -\sigma^2, -\sigma^3)$.
The equations are linear and translation invariant, and can be solved in terms of *plane waves*.

Let's generalize the formalism for spinor fields.
We still want that $\mathcal{L}$ is real and scalar.
For the L representation we can start from the scalar expression

$$\mathcal{L}_L = \Psi_L^+ i\sigma^\mu \partial_\mu \Psi_L \tag{161}$$

Th Euler-Lagrange equations are derived assuming Ψ^+ and Ψ are independent.

$$\partial_\mu \frac{\delta \mathcal{L}_L}{\delta \partial_\mu \Psi_L^+} = \frac{\delta \mathcal{L}_L}{\delta \Psi_L^+} \quad \Rightarrow \quad 0 = i\sigma^\mu \partial_\mu \Psi_L \tag{162}$$

$$\partial_\mu \frac{\delta \mathcal{L}_L}{\delta \partial_\mu \Psi_L} = \frac{\delta \mathcal{L}_L}{\delta \Psi_L} \quad \Rightarrow \quad \partial_\mu \Psi_L^+ i\sigma^\mu = 0 \ \Rightarrow \ (i\sigma^\mu \partial_\mu \Psi_L)^+ = 0 \tag{163}$$

For right handed spinors we have

$$\mathcal{L}_R = \Psi_R^+ i\bar{\sigma}^\mu \partial_\mu \Psi_R \tag{164}$$

For simplicity let's select the wave vector in the 3rd direction. Then our ansatz will take the form:

$$\Psi_L = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-i\omega t - ik_3 x^3} . \tag{165}$$

Inserting this into the equation of motion:

$$(i\sigma^0\partial_0 + i\sigma^3\partial_3]\Psi_L = \begin{pmatrix} \omega + k_3 & 0 \\ 0 & \omega - k_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-i\omega t - ik_3 x^3} = 0 \tag{166}$$

Here we find a linear algebra problem: an eigenvalue equation for ω.
There are two solutions: $\omega = \pm k_3$.
If we accept that the energy is the eigenvalue of the time translation operator, then $E = \omega$ (or $E = \hbar\omega$ if we reintroduce the constants that set our units).
For the case: $\omega > 0$

$$+k_3 \text{ momentum} \qquad \Psi_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\omega t - ik_3 x^3} \tag{167}$$

$$-k_3 \text{ momentum} \qquad \Psi_L = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\omega t + ik_3 x^3} \tag{168}$$

The angular momentum operator (in the z direction) is the matrix

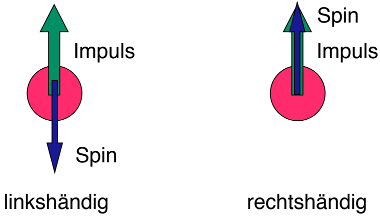
$$J_3 = \frac{1}{2}\sigma^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{169}$$

its effect on the solutions:

$$+k_3 \text{ momentum} \qquad J_3\Psi_L = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\omega t - ik_3 x^3} = -\frac{1}{2}\Psi_L$$

$$-k_3 \text{ momentum} \qquad J_3\Psi_L = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\omega t + ik_3 x^3} = +\frac{1}{2}\Psi_L$$

Helicity: the angular momentum projected on the direction of the momentum.



Ψ_L (at $\omega > 0$) is left handed

Let is now fix $k_3 > 0$. Then we can characterize the solutions by ω:

$$\omega > 0 \qquad J_3\Psi_L = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\omega t - i\omega x^3} = -\frac{1}{2}\Psi_L \qquad (170)$$

$$\omega < 0 \qquad J_3\Psi_L = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\omega t + i\omega x^3} = +\frac{1}{2}\Psi_L \qquad (171)$$

We call the solution with positive energy ($\omega > 0$) as particle, it is left handed.
But there is now a right handed solution, too, if $\omega < 0$. In quantum mechanics we did not have particles with negative energy. We call these *anti-particles*
Analogously in the R representation

$$\omega > 0 \qquad J_3\Psi_R = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\omega t - i\omega x^3} = +\frac{1}{2}\Psi_R \qquad (172)$$

$$\omega < 0 \qquad J_3\Psi_R = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\omega t + i\omega x^3} = -\frac{1}{2}\Psi_R \qquad (173)$$

If we fix our coordinate system to $\vec{k}$, such that $k_3 > 0$ then the two solutions in each representation read:

$$\text{Particle} \qquad \Psi_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ik_3 t - ik_3 x^3} \qquad (174)$$

$$\text{Anti} - \text{particle} \qquad \Psi_L = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{+ik_3 t - ik_3 x^3} \qquad (175)$$

$$\text{Particle} \qquad \Psi_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ik_3 t - ik_3 x^3} \qquad (176)$$

$$\text{Anti} - \text{particle} \qquad \Psi_R = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{+ik_3 t - ik_3 x^3} \qquad (177)$$

What happens now, if we do a spatial reflection?
Spin is a pseudovektor, momentum is a vektor. $\rightarrow$ helicity must be a pseudoscalar: it changes sign with a reflection. **Thus, spatial reflection makes from a Ψ_L particle a Ψ_R particle and a Ψ_L anti-particle is turned into a Ψ_R anti-particle.**
(Note that after a spatial reflection $e^{-i\omega t}$ remains unchanged, so a positive energy will not become negative energy.)

Parity transformation is the spatial reflection with respect to all coordinates (spatial inversion).

Is parity a symmetry of Nature?

If parity is a symmetry then L and R particles must behave the same way. However, e.g. left and right handed neutrinos have very different masses. (In the original Standard Model left handed neutrinos has zero, right handed neutrinos have infinite mass (they don't exist).)

But parity is a symmetry of quantum electrodynamics and quantum chromodynamics.

What equations of motion follow from parity? $\Psi_L \leftrightarrow \Psi_R$.

We build a spinor where the two irreducible representations (L and R) appear in a direct sum. We introduce the *bispinor* with four components:

$$\Psi \equiv \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} \tag{178}$$

Parity transformation:

$$\Psi \to \Psi^P = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} = \begin{pmatrix} 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \Psi = \gamma^0 \Psi \tag{179}$$

Here we defined the first gamma matrix γ^0.

We also introduce a pair of projectors that erase one or the other part of the bispinor:

$$P_L = \frac{1}{2}(1-\gamma^5) \qquad P_L \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} \Psi_L \\ 0 \end{pmatrix} \tag{180}$$

$$P_R = \frac{1}{2}(1+\gamma^5) \qquad P_R \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_R \end{pmatrix} \tag{181}$$

This then defines a new 4×4 gamma matrix:

$$\gamma^5 = \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{182}$$

P_L and P_R are, indeed, projectors since they fulfil

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0\,, \tag{183}$$

where we used that $(\gamma^5)^2 = \mathbf{1}$.

In order to find a generic equtaion of motion we can study the possible Lorentz invariant expressions. We have arleady built the scalars previously: $\Psi_L^+\sigma^\mu\partial_\mu\Psi_L$ and $\Psi_R^+\bar{\sigma}^\mu\partial_\mu\Psi_R$.
Their sum is also a scalar, in the bispinor notation it is

$$i\Psi^+\begin{pmatrix} \sigma^\mu & 0 \\ 0 & \bar{\sigma}^\mu \end{pmatrix}\partial_\mu\Psi \tag{184}$$

For a frequently used combination we intorduce a new symbol:

$$\bar{\Psi} = \Psi^+\gamma^0 = \left(\Psi_R^+, \Psi_L^+\right) \tag{185}$$

Then the scalar (184) has the form:

$$i\bar{\Psi}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\partial_\mu\Psi \tag{186}$$

Now we are in the position to define further *gamma matrices*:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{187}$$

This definition of γ^0 is equivalent to the earlier definition in Eq. (179).
Thus the scalar built from bispinor reads:

$$i\bar{\Psi}\gamma^\mu\partial_\mu\Psi \tag{188}$$

The index μ of γ^μ behaves as a normal four-index.

There is an other scalar, which is bilinear in the fields:

$$\Psi_R^+ \Psi_L + \Psi_L^+ \Psi_R \tag{189}$$

(see Eq. (153)). This, too, can be written in terms of bispinors:

$$\Psi^+ \gamma^0 \Psi = \bar{\Psi}\Psi \tag{190}$$

We can thus combine the two scalars with a free parameter m into one generic scalar:

$$i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi = \bar{\Psi}\left[i\gamma^\mu \partial_\mu - m\right]\Psi \tag{191}$$

This scalar can serve as a Lagrangian. The **Dirac equation** is the corresponding Euler-Lagrange equation:

$$\left[i\gamma^\mu \partial_\mu - m\right]\Psi = 0 \tag{192}$$

It describes a left-handed and a right-handed spinor, that can be transformed between one another. This oscillation between left and right handed spinors is slow if m is small.

The gamma matrices work in the spinor index space, they implement spin and antiparticles. Their algebra can be summarized in two lines:

- $\gamma^5 = \frac{i}{4!}\epsilon_{\alpha\beta\rho\sigma}\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3$
- $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

These follow from the explicit matrix form that we introduced:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \quad (193)$$

This is the *chiral basis* of gamma matrices.
However, it is possible to write down an other set of matrices that fulfil the algebra, e.g. the *Dirac basis*:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad (194)$$

Or the purely imaginary *Majorana basis*:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

All three sets have an additional feature:

$$\left(\gamma^0\right)^+ = \gamma^0 \quad \left(\gamma^j\right)^+ = -\gamma^j \;\; (j = 1\ldots 3) \quad \Rightarrow \gamma^{\mu+}\gamma^0 = \gamma^0\gamma^\mu \quad (195)$$

Independently on which basis (representation) we use the following identities follow from the algebra:

- $\gamma^\mu \gamma^\nu g_{\mu\nu} = 4$
- $\gamma^\mu \gamma^\rho \gamma^\nu g_{\mu\nu} = -2\gamma^\rho$
- $\gamma^\mu \gamma^\rho \gamma^\sigma \gamma^\nu g_{\mu\nu} = 4g^{\rho\sigma}$

The following trace identities are used when the spinor components are summed up (e.g. unpolarized decay of a fermion):

- $\mathrm{Tr}\, \gamma^\mu = 0$
- $\mathrm{Tr}\, \gamma^\mu \gamma^\nu = 4g^{\mu\nu}$
- $\mathrm{Tr}\, \gamma^\mu \gamma^\nu \gamma^\rho = 0$
- $\mathrm{Tr}\, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$

Now we write the Lorentz transformation in bispinor form:

$$S(\Lambda)\Psi = S(\lambda)\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} \Lambda_L\Psi_L \\ \Lambda_R\Psi_R \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}\vec{\sigma}(\vec{\omega}-i\vec{\nu})} & 0 \\ 0 & e^{\frac{i}{2}\vec{\sigma}(\vec{\omega}-i\vec{\nu})} \end{pmatrix}\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} \tag{196}$$

We are after the $S_{\mu\nu}$ antisymmetric tensor, such that

$$\alpha_{\mu\nu} = \begin{pmatrix} 0 & -\nu_x & -\nu_y & -\nu_z \\ \nu_x & 0 & -\omega_z & \omega_y \\ \nu_y & \omega_z & 0 & -\omega_x \\ \nu_z & -\omega_y & \omega_x & 0 \end{pmatrix} \tag{197}$$

$$S(\Lambda)\Psi = 1 - \frac{i}{2}\alpha_{\mu\nu}S^{\mu\nu}\Psi + \mathcal{O}(\alpha^2) \tag{198}$$

Of course $S_{\mu\nu}$ works on the bispinor indices, that we want to build from the γ matrices. There is only this choice for an antisymmetric tensor:

$$\frac{1}{4}[\gamma^\mu,\gamma^\nu]\alpha_{\mu\nu} = -\frac{1}{2}(\gamma^0\gamma^a - \gamma^a\gamma^0)\nu_a - \frac{1}{4}\epsilon_{abc}[\gamma^a,\gamma^b]\omega_c \tag{199}$$

We can write this combination explicitely in the chiral basis:

$$\begin{aligned}
\gamma^0\gamma^j - \gamma^j\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} - \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} = 2\begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \qquad (200) \\
\gamma^a\gamma^b - \gamma^b\gamma^a &= \begin{pmatrix} -\sigma^a\sigma^b & 0 \\ 0 & -\sigma^a\sigma^b \end{pmatrix} - \begin{pmatrix} -\sigma^b\sigma^a & 0 \\ 0 & -\sigma^b\sigma^a \end{pmatrix} \\
&= -2i\epsilon_{abd}\begin{pmatrix} \sigma^d & 0 \\ 0 & \sigma^d \end{pmatrix} \qquad (201)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4}[\gamma^\mu, \gamma^\nu]\alpha_{\mu\nu} &= -\frac{1}{2}(\gamma^0\gamma^a - \gamma^a\gamma^0)\nu_a - \frac{1}{4}\epsilon_{abc}[\gamma^a, \gamma^b]\omega_c \qquad (202) \\
&= \begin{pmatrix} \vec{\sigma}\vec{\nu} & 0 \\ 0 & -\vec{\sigma}\vec{\nu} \end{pmatrix} + \frac{i}{2}\epsilon_{abd}\epsilon_{abc}\begin{pmatrix} \sigma^d & 0 \\ 0 & \sigma^d \end{pmatrix}\omega_c \\
&= i\begin{pmatrix} (\vec{\omega} - i\vec{\nu})\vec{\sigma} & 0 \\ 0 & (\vec{\omega} + i\vec{\nu})\vec{\sigma} \end{pmatrix} \qquad (203)
\end{aligned}$$

Thus, $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ is

$$S(\Lambda)\Psi = \Psi - \frac{i}{2}\alpha_{\mu\nu}S^{\mu\nu}\Psi + \mathcal{O}(\alpha^2) \qquad (204)$$

How to contract bispinor indices? Let's have two bispniros ψ and η. Then there are four covariant combinations without an outer bispinor index.

$$\bar{\psi}\eta \qquad \text{scalar} \tag{205}$$

$$\bar{\psi}\gamma^{\mu}\eta \qquad \text{four} - \text{vector} \tag{206}$$

$$\bar{\psi}\sigma^{\mu\nu}\eta \qquad \text{antisymmetrictensor} \tag{207}$$

$$\bar{\psi}\gamma^{\mu}\gamma^{5}\eta \qquad \text{axialvector} \tag{208}$$

$$\bar{\psi}\gamma^{5}\eta \qquad \text{pseudoscalar} \tag{209}$$

Mit $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}\gamma^{\nu}]$.

The Dirac equation is not the only covariant equation that can be formed.

$$[i\gamma^{\mu}\partial_{\mu} + m][i\gamma^{\nu}\partial_{\nu} - m]\Phi = 0 \tag{210}$$

Following the algebraic rules we get

$$[-g^{\mu\nu}\partial_{\mu}\partial_{\nu} - m^2]\Phi = 0\,. \tag{211}$$

This equation is a set of four independent copies of the Klein-Gordon equation.

The free dirac equation can be solved with a plane wave ansatz:

$$i(\gamma^\mu \partial_\mu - m)\Psi = 0 \tag{212}$$

$$\Psi_p = u_p e^{-ip_\mu x^\mu} + v_p e^{ip_\mu x^\mu} \tag{213}$$

where u_p and v_p are constant bispinors. We immedately find

$$(p_\mu \gamma^\mu - m)u_p = 0, \qquad (p_\mu \gamma^\mu + m)v_p = 0\,, \tag{214}$$

or with the *Feynman slash notation* :

$$\sum_\mu \partial_\mu \gamma^\mu = \not{\partial}, \qquad \sum_\mu p_\mu \gamma^\mu = \not{p} \tag{215}$$

Thus, e.g.

$$\not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu = p_\mu p_\nu \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = p_\mu p_\nu g^{\mu\nu} = p^2 \tag{216}$$

And then we get

$$(\not{p} - m)u_p = 0, \qquad (\not{p} + m)v_p = 0\,. \tag{217}$$

If we select a basis, e.g. the Dirac basis

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{218}$$

then we can write down the matrix of the eigenvalue problem:

$$\begin{pmatrix} p_0 - m & \vec{p}\vec{\sigma} \\ -\vec{p}\vec{\sigma} & -p_0 - m \end{pmatrix} u_p = 0 \tag{219}$$

A zero determinant is searched for. Since the determinant is independent of the coordinate system, we pick our choice: the 3rd axis is in the direction of $\vec{p}$.

$$\begin{aligned} 0 &= \begin{vmatrix} p_0 - m & \vec{p}\vec{\sigma} \\ -\vec{p}\vec{\sigma} & -p_0 - m \end{vmatrix} = \begin{vmatrix} p_0 - m & 0 & |\vec{p}| & 0 \\ 0 & p_0 - m & 0 & -|\vec{p}| \\ -|\vec{p}| & 0 & -p_0 - m & 0 \\ 0 & |\vec{p}| & 0 & -p_0 - m \end{vmatrix} \\ &= \left(p_0^2 - |\vec{p}|^2 - m^2\right)^2 = \left(p^2 - m^2\right)^2 \\ \rightarrow \quad & m \equiv \text{ rest mass} \end{aligned} \tag{220}$$

$$\begin{pmatrix} p_0 - m & \vec{p}\vec{\sigma} \\ -\vec{p}\vec{\sigma} & -p_0 - m \end{pmatrix} u_p = 0 \qquad \begin{pmatrix} p_0 + m & \vec{p}\vec{\sigma} \\ -\vec{p}\vec{\sigma} & -p_0 + m \end{pmatrix} v_p = 0 \tag{221}$$

We introduce the shorthand $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$.
In both (u_p and v_p) cases the determinant is zero if $p_0 = \pm\omega_p$. Für $p_0 = \omega_p$:

$$\begin{pmatrix} \omega_p - m & \vec{p}\vec{\sigma} \\ -\vec{p}\vec{\sigma} & -\omega_p - m \end{pmatrix} u_p = 0 \qquad \begin{pmatrix} \omega_p + m & \vec{p}\vec{\sigma} \\ -\vec{p}\vec{\sigma} & -\omega_p + m \end{pmatrix} v_p = 0 \tag{222}$$

Formal solution for u_p

$$\begin{pmatrix} \omega_p - m & \vec{p}\vec{\sigma} \\ -\vec{p}\vec{\sigma} & -\omega_p - m \end{pmatrix} \begin{pmatrix} \omega_p + m \\ \vec{p}\sigma \end{pmatrix} = 0 \tag{223}$$

This product is a 4×2 matrix. We multiply from right with a χ_s 2×1 matrix ($\equiv$ a spinor):

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{224}$$

After normalization we have: $u_p = \sqrt{\omega_p + m} \begin{pmatrix} \chi_s \\ \frac{\vec{p}\vec{\sigma}}{\omega_p + m}\chi_s \end{pmatrix}$.

So there are four solutions in total:

$$u(p,s) = \sqrt{\omega_p + m}\begin{pmatrix} \chi_s \\ \frac{\vec{p}\vec{\sigma}}{\omega_p+m}\chi_s \end{pmatrix} \tag{225}$$

$$v(p,s) = \sqrt{\omega_p + m}\begin{pmatrix} \frac{\vec{p}\vec{\sigma}}{\omega_p+m}\chi_s \\ \chi_s \end{pmatrix} \tag{226}$$

$$\tag{227}$$

with $s = 1, 2$. The normalization was chosen such that

$$\bar{u}(p,s)u(p,s') = 2m\delta_{s,s'} , \qquad \bar{v}(p,s)v(p,s') = -2m\delta_{s,s'} . \tag{228}$$

Then the dyadic product, too, has a closed form (*completeness relation*)

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = (\not{p} + m)_{\alpha\beta} \tag{229}$$

$$\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = (\not{p} - m)_{\alpha\beta} \tag{230}$$

From the Dirac equation one can construct a *conserved charge*:
Let us multiply the Dirac equation by $\bar{\Psi}$ from left:

$$\bar{\Psi}\left[i\gamma^{\mu}\partial_{\mu}-m\right]\Psi=\bar{\Psi}[\partial_{\mu}i\gamma^{\mu}\Psi]-m\bar{\Psi}\Psi=0 \tag{231}$$

Or one can take the adjoint of the Dirac equation

$$\partial_{\mu}\Psi^{+}(-i)\gamma^{\mu+}-m\Psi^{+}=0 \tag{232}$$

and multiply it by $\gamma^{0}\Psi$ from right:

$$[\partial_{\mu}\Psi^{+}](-i)\gamma^{\mu+}\gamma^{0}\Psi-m\Psi^{+}\gamma^{0}\Psi=0 \tag{233}$$

We have selected the γ matrices such that $\gamma^{\mu+}\gamma^{0}=\gamma^{0}\gamma^{\mu}$, see Eq. (195):

$$[\partial_{\mu}\bar{\Psi}](-i\gamma^{\mu})\Psi-m\bar{\Psi}\Psi=0 \tag{234}$$

Subtracting Eq. (234) from (231) we have:

$$\bar{\Psi}[\partial_{\mu}i\gamma^{\mu}\Psi]+[\partial_{\mu}\bar{\Psi}]i\gamma^{\mu}\Psi=\partial_{\mu}[\bar{\Psi}\gamma^{\mu}\Psi]=0 \tag{235}$$

$$\text{for } j^{\mu}=\bar{\Psi}\gamma^{\mu}\Psi\ :\quad \partial_{\mu}j^{\mu}=0\equiv\frac{\partial j^{0}}{\partial t}+\vec{\nabla}\vec{j}=0\quad\Rightarrow\quad\int d^{3}r\ \Psi^{+}\Psi=\text{const} \tag{236}$$

Can we calculate the charge density j^0 for the solutions u_p and v_p?

$$\Psi_p = u_p e^{-ip_\mu x^\mu} + v_p e^{ip_\mu x^\mu} \tag{237}$$

u_p describes a field with energy $p_0 > 0$ and momentum $\vec{p}$, and analogously v_p describes a field with energy $-p_0$ and momentum $-\vec{p}$.

For the u_p solution (averaging over the two possible χ_s vectors), see Eq. (229):

$$\frac{1}{2}\sum_s \bar{u}_{p,s}\gamma^\mu u_{p,s} = \frac{1}{2}\sum_s \sum_{\alpha,\beta} \bar{u}_{p,s\,\alpha}\gamma^\mu_{\alpha\beta} u_{p,s\,\beta} = \frac{1}{2}\sum_s \sum_{\alpha,\beta} (\not{p} + m)_{\beta\alpha}\gamma^\mu_{\alpha\beta} \tag{238}$$

$$\frac{1}{2}\sum_s \bar{u}_{p,s}\gamma^\mu u_{p,s} = \frac{1}{2}\mathrm{Tr}\,(\not{p} + m)\gamma^\mu = \frac{1}{2}\mathrm{Tr},\, p_\nu\gamma^\nu\gamma^\mu + m\,\frac{1}{2}\mathrm{Tr}\,\gamma^\mu = 2p^\mu \tag{239}$$

Similarly we get for the v solution:

$$\frac{1}{2}\sum_s \bar{v}_{p,s}\gamma^\mu v_{p,s} = \frac{1}{2}\mathrm{Tr}\,(\not{p} - m)\gamma^0 = \frac{1}{2}\mathrm{Tr},\, p_\nu\gamma^\nu\gamma^\mu - m\,\frac{1}{2}\mathrm{Tr}\,\gamma^\mu = 2p^\mu \tag{240}$$

For the u solution $\vec{j}$ is in the direction of the momentum, but for v $\vec{j}$ and $\vec{p}$ are pointing oppositely.

Charge conjugation: Suppose in Nature particles and anti-particles behave the same way and if all particles are turned into anti-particles and vice versa, then the laws of Nature stay unchanged. This would be charge conjugation symmetry.
Charge conjugation is realized by the following linear transformation:

$$\psi^{(c)} = C\psi^* \tag{241}$$

where C is a 4×4 matrix with the following features

$$C^+ C = 1 \qquad \text{and} \qquad C^+ \gamma^\mu C = -(\gamma^\mu)^* \quad \mu = 0 \ldots 3 \tag{242}$$

e.g. in the chiral basis we can use $C = i\gamma^2$. In the Dirac base one would use $C = i\gamma^2\gamma^0$.
Statement: The Lorentz transformation of a charge conjugated field is the charge conjugation of the Lorentz transformed field.

$$S[\Lambda]\psi^{(c)} = S[\Lambda]C\psi^* = CS[\Lambda]^*\psi^* = C\,(S[\Lambda]\psi)^* = (S[\Lambda]\psi)^{(c)} \tag{243}$$

Here S is the spinor representation of the Lorentz group. For an infinitesimal transformation:

$$S(\Lambda(\alpha)) = 1 - \frac{i}{2}\alpha_{\mu\nu}S^{\mu\nu} \quad \text{and} \quad S^{\mu\nu} = \frac{i}{4}\,[\gamma^\mu, \gamma^\nu] \tag{244}$$

Now we can write down the charge conjugated Dirac equation:

$$\begin{aligned} (i\partial\!\!\!/ - m)\psi = 0 \quad &\Rightarrow \quad (-i\partial\!\!\!/^* - m)\psi^* = 0 \\ &\Rightarrow \quad C(-i\partial\!\!\!/^* - m)\psi^* = (i\partial\!\!\!/ - m)\psi^{(c)} = 0 \end{aligned} \tag{245}$$

The Dirac equation is symmetric under charge conjugation.
How do the solutions behave?

$$\text{In Dirac basis}: \; C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \tag{246}$$

Let's have a u_p solution, first:

$$\begin{aligned} Cu_p e^{-ipx} &= \sqrt{\omega_p + m} C \begin{pmatrix} 1 \\ \frac{\vec{p}\vec{\sigma}}{m+\omega_p} \end{pmatrix} \chi_s e^{-ipx} = \sqrt{\omega_p + m} \begin{pmatrix} -i\sigma^2 \frac{\vec{p}\vec{\sigma}}{m+\omega_p} \\ -i\sigma^2 \end{pmatrix} \chi_s e^{ipx} \\ &= \sqrt{\omega_p + m} \begin{pmatrix} -\frac{\vec{p}\vec{\sigma}}{m+\omega_p} \\ 1 \end{pmatrix} \underbrace{(-i\sigma^2)\chi_s}_{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \chi_s} e^{ipx} \end{aligned} \tag{247}$$

With $q^\mu = (p^0, -p^1, -p^2, -p^3)$: $v(q,s) = \sqrt{\omega_q + m} \begin{pmatrix} \frac{\vec{q}\vec{\sigma}}{\omega_q+m} \\ 1 \end{pmatrix} \chi_s$.

(The matrix in front of χ_s is a -180^o rotation $e^{-i\frac{\sigma_2}{2}\pi} = -i\sigma_2$)

Example scalar field and translation

$$\begin{aligned} x^\mu &\rightarrow x^\mu - \varepsilon a^\mu \qquad a^\mu \text{ is a constant vector} && (248)\\ \phi(x) &\rightarrow \phi(x+\varepsilon a) = \phi(x) + \varepsilon a^\mu \partial_\mu \phi(x) && (249)\\ \mathcal{L}(x) &\rightarrow \mathcal{L}(x+\varepsilon a) = \mathcal{L}(x) + \varepsilon a^\mu \partial_\mu \mathcal{L}(x) && (250) \end{aligned}$$

In our case $\Delta\phi = a^\mu \partial_\mu \phi$ and $\Delta\mathcal{L} = a^\mu \partial_\mu \mathcal{L}$ and, thus, $J^\mu = a^\mu \mathcal{L}$

$$j^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \Delta\Phi - J^\mu = a^\nu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - a^\mu \mathcal{L} \qquad (251)$$

a^ν can be any vector if a symmetry is valid in all directions. Let us consider all four possible basis vectors in the canonical basis: $a^\nu = \delta^\nu_\rho$:

$$\begin{aligned} j^\mu_{(\nu)} &= \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} =: T^\mu{}_\nu \quad \textit{Energy momentum tensor} && (252)\\ \partial_\mu T^\mu{}_\nu(x) &= 0 && (253)\\ Q_{(\rho)} &= \int d^3x \left(\frac{\partial \mathcal{L}}{\delta(\partial_0 \phi)} \partial_\rho \phi - \delta^0_\rho \mathcal{L} \right) && (254) \end{aligned}$$

$\Pi = \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi)}$ is called the *canonical momentum*. In the conserved charge we recognize the *Hamilton function*, which gives the energy

$$Q_{(0)} = \int d^3x \left(\Pi(x)\dot{\phi}(x) - \mathcal{L} \right) = \int d^3x \mathcal{H}(\Pi, \phi) \qquad (255)$$

Example: Ψ shall be a complex scalar field. The Lagrangian must be real and scalar. We wish to form an action with charge conjugation symmetry.

$$C\Phi(x) = \Phi^*(x)\,, \qquad \mathcal{L}[C\Phi(x)] = (\mathcal{L}[\Phi(x)])^* = \mathcal{L}[\Phi(x)] \tag{256}$$

The simplest form with the desired symmetries:

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi \tag{257}$$

Let us derive the equations of motion:
For the derivation we can pretend Φ and Φ^* to be independent fields. We use then the generic formula for the E-L equations:

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi^*} - \frac{\delta \mathcal{L}}{\delta \Phi^*} = 0 \quad \Rightarrow \quad \partial_\mu \partial^\mu \Phi - m^2 \Phi = 0 \tag{258}$$

$$\partial^\mu \frac{\delta \mathcal{L}}{\delta \partial^\mu \Phi} - \frac{\delta \mathcal{L}}{\delta \Phi} = 0 \quad \Rightarrow \quad \partial_\mu \partial^\mu \Phi^* - m^2 \Phi^* = 0 \tag{259}$$

Alternatively, with $\Phi_1 = \sqrt{2}\mathrm{Re}\,\Phi$ and $\Phi_2 = \sqrt{2}\mathrm{im}\,\Phi$ we can write $\mathcal{L}$ as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \Phi_1)(\partial^\mu \Phi_1) + \frac{1}{2}(\partial_\mu \Phi_2)(\partial^\mu \Phi_2) - \frac{1}{2}m^2(\Phi_1)^2 - \frac{1}{2}m^2(\Phi_2)^2 \\ &= \frac{1}{2}\sum_j (\partial_\mu \Phi_j)(\partial^\mu \Phi_j) - \frac{1}{2}\sum_j m^2 \Phi_j \Phi_j \end{aligned} \tag{260}$$

Let's have a closer look on the symmetry:

$$\Phi \rightarrow e^{i\varepsilon}\Phi\,, \quad \Phi^* \rightarrow \Phi^* e^{-i\varepsilon} \tag{261}$$

For infinitesimal transformations:

$$\Delta\Phi = i\varepsilon\Phi\,, \quad \Delta\Phi^* = -i\varepsilon\Phi^*\,, \quad \Delta\mathcal{L} = 0 \tag{262}$$

This gives the following Noether current:

$$j^\mu\varepsilon = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi)}\Delta\Phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi^*)}\Delta\Phi^* = i(\Phi\partial_\mu\Phi^* - \Phi^*\partial_\mu\Phi)\varepsilon \tag{263}$$

and the conserved charge:

$$Q = \int d^3x i(\Phi\partial_0\Phi^* - \Phi^*\partial_0\Phi) \tag{264}$$

We could have worked with Φ_1 and Φ_2 fields just as well:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \tag{265}$$

$$j^\mu\varepsilon = \sum_j \frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi_j)}\Delta\Phi_j = (\partial^\mu\Phi_1, \partial^\mu\Phi_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \varepsilon \tag{266}$$

The Lagrangian can be more complicated keeping the same symmetries. We only need to keep in mind that we should use scalars in the internal space: i.e. the scalar product $\sum_j \Phi_j\Phi_j = |\Phi|^2$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_j)(\partial^\mu \Phi_j) - V(\sqrt{\Phi_j\Phi_j})\,, \qquad V(\varrho) = \frac{1}{2}\alpha\varrho^2 + \frac{1}{24}\varrho^4 \tag{267}$$

The conserved charge Q vill not depend on the potential V. In general

$$Q = \int d^3x(\partial_0\Phi)T\Phi\,, \qquad T : \text{Generator of the symmetry, e.g.} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The internal symmetry group may be larger and the index may go from $1 \dots N$. That would correspond to an invariance under the rotations in the $O(N)$ group, or the $U(N)$ group if there are several complex components:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_j^*)(\partial^\mu \Phi_j) - V(\sqrt{\Phi_j^*\Phi_j}) \tag{268}$$

The argument of Noether's theorem can be repeated with every generator of the symmetry. Thus there are as many conserved charges as generators in the Lie group.

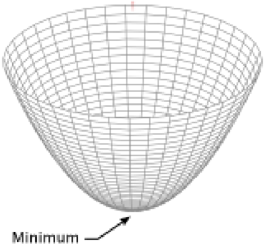
Note that here we talked about a new internal symmetry, which is not included in the Lorentz group.

In the previous example (complex scalar field) the α parameter plays an important role:

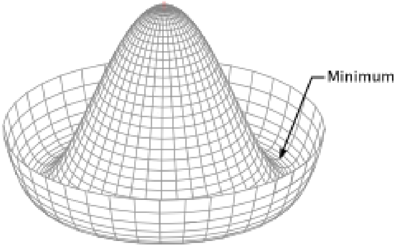
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_j)(\partial^\mu \Phi_j) - V(\sqrt{\Phi_j \Phi_j}) , \qquad V(\varrho) = \frac{1}{2}\alpha\varrho^2 + \frac{1}{24}\varrho^4 \tag{269}$$

The form of the potential depends on the sign of α:

$\alpha > 0$



$\alpha < 0$



For the field Φ one can define a homogeneous part $\bar{\Psi}(t)$

$$\Phi(x) = \bar{\Phi}(t) + \phi(t,x) \qquad \int d^3\phi(x) = 0 \tag{270}$$

which is moving in this potential.

If $\alpha > 0$ there is a well defined symmetric minimum, and $\sqrt{\alpha}$ sets the mass.

If, however, $\alpha < 0$ the *ground state* is not at $\Phi \equiv 0$, sondern:

- There are several ground states (field configurations with lowest energy). The symmetry transformation can be used to transform one such ground state into the other.
- If the system is in one of the ground states, that is, the dynamics of the system has selected one minimum, the symmetry is *spontaneously broken*. The Lagrangian is completely symmetric.

In the simplest case the system is in the ground state $\bar{\Phi} = \begin{pmatrix} v \\ 0 \end{pmatrix}$. Then we can introduce a shifted field $\Phi(x) = \bar{\Phi} + \phi(x)$, and with this, the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_j)(\partial^\mu \phi_j) - V \quad V = \frac{1}{2}\alpha\left((\phi_1 + v)^2 + \phi_2^2\right) + \frac{1}{24}\lambda\left((\phi_1 + v)^2 + \phi_2^2\right)^2 \tag{271}$$

Since $\bar{\Phi}$ is a ground state, v must be such that $\partial V / \partial \phi_1|_{\phi \equiv = 0} = 0$.

$$\left.\frac{\partial V}{\partial \phi_1}\right|_{\phi\equiv 0} = \alpha v + \frac{\lambda}{6} v^3 = 0 \qquad \left.\frac{\partial V}{\partial \phi_2}\right|_{\phi\equiv 0} = 0 \tag{272}$$

We can linearize the field equations around the minimum $\bar{\Phi}$ of the potential. Remember, that for a free field the mass is the second derivative with respect to the potential (the first derivative is zero).

$$V = \frac{1}{2} m^2 \phi^2 \quad \Rightarrow \quad m^2 = \left. \frac{\partial^2 V}{(\partial \phi)^2} \right|_{\phi \equiv 0} \tag{273}$$

In the presence of several field components the second derivative is a matrix and there may be off-diagonal elements. With an appropriate diagonalization one can select eigenvectors, which will be the *mass eigenstates*.
In our case:

$$\left. \begin{pmatrix} \frac{\partial^2 V}{\partial \phi_1 \partial \phi_1} & \frac{\partial^2 V}{\partial \phi_1 \partial \phi_2} \\ \frac{\partial^2 V}{\partial \phi_2 \partial \phi_1} & \frac{\partial^2 V}{\partial \phi_2 \partial \phi_2} \end{pmatrix} \right|_{\phi \equiv 0} = \begin{pmatrix} -2\alpha & 0 \\ 0 & 0 \end{pmatrix} \tag{274}$$

Here we simplified the component $\frac{\partial^2 V}{\partial \phi_1 \partial \phi_1} = \alpha - \frac{\lambda}{24} v^2$ using $\partial V / \partial \phi_1 = \alpha v + \frac{\lambda}{6} v^3 = 0$. The masses^2 are the eigenvalues of the matrix

$$\text{radial } m_R^2 = -2\alpha \,, \qquad \text{symmetry} \equiv \text{Goldstone } m_G^2 = 0 \tag{275}$$

Goldstone theorem: For each broken continuous symmetry there is a mass eigenstate with zero eigenvalue. Thus, there shall be a particle field with zero mass (Nambu-Goldstone boson).

Now putting the R and L spinors in a bispinor: $\Psi \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_L + \mathcal{L}_R && (276)\\
&= i\Psi^+ \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \bar{\sigma}^\mu \end{pmatrix} \partial_\mu \Psi && (277)\\
&= i\Psi^+ \gamma^0 \gamma^\mu \partial_\mu \Psi \qquad \boxed{\bar{\Psi} = \Psi^+ \gamma^0} && (278)\\
&= i\bar{\Psi}\gamma^\mu \partial_\mu \Psi && (279)
\end{aligned}$$

Now we are working in the chiral basis of the γ matrices.
The other scalar expression:

$$\Psi_R^+ \Psi_L + \Psi_L^+ \Psi_R = (\Psi_L^+, \Psi_R^+)\, \gamma^0 \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \Psi^+ \gamma^0 \Psi = \bar{\Psi}\Psi \tag{280}$$

since γ^0 has been constructed to exchange the R and L spinors

$$\begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} = \gamma^0 \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} . \tag{281}$$

Both terms together give the Dirac Lagrangian

$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{282}$$

Let us calculate the equation of motion from

$$\mathcal{L}_D = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{283}$$

Again, let's take the functional derivatives with respect to $\bar{\Psi}$ and Ψ as independent variables.

$$\partial_\mu \frac{\delta \mathcal{L}_D}{\delta \partial_\mu \bar{\Psi}} = \frac{\delta \mathcal{L}_D}{\delta \bar{\Psi}} \quad \Rightarrow \quad 0 = (i\partial\!\!\!/ - m)\Psi \tag{284}$$

$$\partial_\mu \frac{\delta \mathcal{L}_D}{\delta \partial_\mu \Psi} = \frac{\delta \mathcal{L}_D}{\delta \Psi} \quad \Rightarrow \quad i\partial_\nu \bar{\Psi}\gamma^\nu = -m\bar{\Psi} \tag{285}$$

The second equation can be brought to a better known from:

$$-i\partial_\nu \gamma^{\nu +}(\bar{\Psi})^+ + m(\bar{\Psi})^+ = 0 \quad \Rightarrow \quad -(i\partial_\nu \gamma^{\nu +} - m)\gamma^0\Psi = 0 \quad \boxed{(\bar{\Psi})^+ = \gamma^0\Psi}$$

$$\Rightarrow \quad -\gamma^0(i\partial_\nu \gamma^\nu - m)\Psi = 0 \quad \boxed{\gamma^{\mu +}\gamma^0 = \gamma^0\gamma^\mu}$$

Thus from both varians we get the Dirac equation.

All three forms ($\mathcal{L}_L,\ \mathcal{L}_R,\ \mathcal{L}_D$) are invariant under a symmetry transformation:

$$\Psi \to \Psi e^{i\alpha}\,, \quad \bar{\Psi} \to \bar{\Psi} e^{-i\alpha} \tag{286}$$

Both terms of the Dirac Lagrangian are invariant:

$$\mathcal{L}_D = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \quad \to \quad \bar{\Psi}e^{-i\alpha}(i\partial\!\!\!/ - m)e^{i\alpha}\Psi = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{287}$$

For this there exists a Noether current

$$j^\mu \alpha \quad = \quad \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Psi)} \Delta\Psi = i\bar{\Psi}\gamma^\mu i\alpha\Psi \tag{288}$$

$$j^\mu \quad \sim \quad \bar{\Psi}\gamma^\mu\Psi \tag{289}$$

We could introduce an other *(chiral)* transformation:

$$\Psi \to \Psi e^{+i\beta\gamma^5}\,, \quad \bar{\Psi} \to \bar{\Psi} e^{+i\beta\gamma^5} \tag{290}$$

The kinetic term is invariant, and one can define a current.

$$j_5^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi\,, \tag{291}$$

In the massive case, however, the Lagrangian is not invariant, and the four-divergence of the current will be non-zero.

$$\partial_\mu j_5^\mu = 2mi\bar{\Psi}\gamma^5\Psi \tag{292}$$

So far we worked with *global symmetries*. E.g. the global U(1) symmetry a conserved charge can be derived. Dependeng on context, it can be called electric charge or baryon number or lepton number.

$$\Psi \to e^{-i\alpha}\Psi \qquad \bar{\Psi} \to \bar{\Psi}e^{i\alpha} \tag{293}$$

Local symmetry is the invariance under a space-time dependent transformation (e.g. phase rotation):

$$\Psi \to e^{-i\alpha(x)}\Psi \qquad \bar{\Psi} \to \bar{\Psi}e^{i\alpha(x)} \tag{294}$$

$$\begin{aligned}
\mathcal{L} &\to \bar{\Psi}(x)e^{i\alpha(x)}[i\gamma^\mu\partial_\mu - m]e^{-i\alpha(x)}\Psi(x) \\
&\to \bar{\Psi}(x)e^{i\alpha(x)}[i\gamma^\mu\partial_\mu]e^{-i\alpha(x)}\Psi(x) - \bar{\Psi}(x)m\Psi(x) \\
&\to \bar{\Psi}(x)[i\gamma^\mu\partial_\mu - m]\Psi(x) + \bar{\Psi}(x)e^{i\alpha(x)}\left[i\gamma^\mu e^{-i\alpha(x)}(-i)\partial_\mu\alpha(x)\right]\Psi(x) \\
&\to \bar{\Psi}(x)\left[i\gamma^\mu\partial_\mu - m\right]\Psi(x) + \bar{\Psi}(x)\left[\gamma^\mu\partial_\mu\alpha(x)\right]\Psi(x)
\end{aligned} \tag{295}$$

So this Lagrangian is not symmetric under the local transformation.

The Dirac Lagrangian tronsforms under local U(1) as

$$\mathcal{L} \quad \rightarrow \quad \bar{\Psi}(x)\left[i\gamma^{\mu}\partial_{\mu} - m\right]\Psi(x) + \bar{\Psi}(x)\left[\gamma^{\mu}\partial_{\mu}\alpha(x)\right]\Psi(x) \tag{296}$$

Now we can introduce some other terms to the Lagrangian that produce the same term with opposite term under the same local transformation.

$$A_{\mu}(x) \quad \rightarrow \quad A_{\mu}(x) + \frac{1}{e}\partial_{\mu}\alpha(x) \tag{297}$$

$$\bar{\Psi}(x)\gamma^{\mu}A_{\mu}(x)\Psi(x) \quad \rightarrow \quad \frac{1}{e}\bar{\Psi}(x)\gamma^{\mu}[\partial_{\mu}\alpha(x)]\Psi(x) \tag{298}$$

Then we can design a Lagrangan with the local symmetry

$$\mathcal{L} = \bar{\Psi}(x)[i\gamma^{\mu}\partial_{\mu} - m]\Psi(x) - e\bar{\Psi}(x)\gamma^{\mu}A_{\mu}(x)\Psi(x)\,, \tag{299}$$

which we write with the *covariant derivative* $D_{\mu} = \partial_{\mu} + ieA_{\mu}(x)$, as

$$\mathcal{L} = \bar{\Psi}(x)[i\gamma^{\mu}D_{\mu} - m]\Psi(x) \tag{300}$$

If we want to attach a physical meaning to the A_{μ} fields then this physics must also be invariant under Eq. (297).

We introduced the A_μ fields, let us discuss its dynamics. We need extra terms in the Lagrangian so that an equation of motion can be derived. For a non-trivial Euler-Lagrange equation we need the derivatives of A_μ, but without destroying the postulated local $U(1)$ symmetry or the Lorentz symmetry. We will use D_μ instead of A_μ since D_μ is a Lorentz vector and it is *gauge covariant*:

$$\begin{aligned} D_\mu\Psi(x) \quad &\rightarrow \quad [\partial_\mu + ieA_\mu + i(\partial_\mu\alpha(x))]e^{-i\alpha(x)}\Psi(x) \\ &= e^{-i\alpha(x)}[\partial_\mu + ieA_\mu]\Psi(x) = e^{-i\alpha(x)}D_\mu\Psi(x) \end{aligned} \tag{301}$$

Now D_μ is a derivative, using commutators we can make expressions what are not derivatives:

The pattern: $[\partial_x, f(x)]g(x) = \partial_x(f(x)g(x)) - f(x)\partial_x g(x) = (\partial_x f(x))g(x)$

$$\Rightarrow \qquad [\partial_x, f(x)] = (\partial_x f(x))$$

$$\begin{aligned} [D_\mu, D_\nu] \quad &= \quad D_\mu D_\nu - D_\nu D_\mu \\ &= \quad (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\mu) - (\partial_\nu + ieA_\nu)(\partial_\mu + ieA_\mu) \\ &= \quad ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = ieF_{\mu\nu} \end{aligned} \tag{302}$$

Here we could recognise the field strength tensor F.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{303}$$

For the case of the abelian $U(1)$ symmetry $F_{\mu\nu}$ invariant:

$$\begin{aligned} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \rightarrow \quad & \partial_\mu (A_\nu + \frac{1}{e}\partial_\nu \alpha) - \partial_\nu (A_\mu + \frac{1}{e}\partial_\mu \alpha) \\ & = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{e}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\alpha \\ & = F_{\mu\nu} \end{aligned} \tag{304}$$

Let us pause here and check the dimensions. The action has ($\hbar = 1$), or in natural units 1, which is mass to the 0th power. In 4 dimensions follow

$$[S] = 0, \quad [\mathcal{L}] = 4, \quad [\partial_\mu] = 1, \quad [A_\mu] = 1, \quad [F_{\mu\nu}] = 2 \tag{305}$$

The simplest gauge invariant scalars with dimension 4 are

$$F_{\mu\nu}F^{\mu\nu} \qquad D_\mu D_\nu F^{\mu\nu} \tag{306}$$

Now, since F is antisymmetric $D_\mu D_\nu F^{\mu\nu}$ is equivalent to $\frac{1}{2}[D_\mu, D_\nu]F^{\mu\nu}$. The latter is proportional to $F_{\mu\nu}F^{\mu\nu}$.
So we can stay with the gauge invarinat scalar

$$\mathcal{L}_{photon} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \, . \tag{307}$$

A psuedo-scalar is also possible to form:

$$\mathcal{L}_{PS} = c\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} = c\tilde{F}^{\mu\nu}F_{\mu\nu} \tag{308}$$

this induces parity breaking and may play a role in Standard Model extensions.

Gauge invariant Lorentz scalars can be added up and will stay symmetric. Thus we can add the various possible Lagrangians and become a theory that describes the interaction between photons and fermions. This new theory is called *quantum electrodynamics*.

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{309}$$

with

$$\not{D} = \gamma^\mu(\partial_\mu + ieA_\mu) \tag{310}$$

Terms that are not quadratic are called interaction terms. Writing them separately we have free fields + interaction with a coupling constant e:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{e\bar{\Psi}\gamma^\mu A_\mu \Psi}_{\mathcal{L}_{\mathrm{interaction}}} \tag{311}$$

Let us derive the Euler-Lagrange equations:

$$\mathcal{L}_{photon} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{312}$$

$$\partial_\nu \frac{\delta\mathcal{L}_{photon}}{\delta\partial_\nu A_\mu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\nu F^{\mu\nu} \tag{313}$$

$$\frac{\delta\mathcal{L}_{interaction}}{\delta A_\mu} = -e\bar{\psi}\gamma^\mu\psi = -j^\mu \tag{314}$$

Thus for the gauge fields we obtained the equaton of motion

$$\partial_\nu F^{\mu\nu} = -j^\mu \tag{315}$$

The current j^μ is the Noether current. *Actually, the global U(1) symmetry is part of the local symmetry and remains valid in the presence of the gauge fields.*
j^μ is a four-vector with the components: $j^\mu = (\varrho, \vec{j})$. Using Eq. (17)

$$\mu = 0 \quad \Rightarrow \quad \partial_\nu F^{0\nu} = -\vec{\nabla}\vec{E} = -\varrho \qquad\qquad \vec{\nabla}\vec{E} = \varrho \tag{316}$$

$$\mu = 1 \quad \Rightarrow \quad \partial_\nu F^{1\nu} = \dot{E}_x + \underbrace{\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}}_{-(\vec{\nabla}\times\vec{B})_x}$$

$$\mu = k \quad \Rightarrow \quad \partial_\nu F^{k\nu} = \dot{E}_k - (\vec{\nabla}\times\vec{B})_k = -j_k \qquad \vec{\nabla}\times\vec{B} - \dot{\vec{B}} = \vec{j} \tag{317}$$

The Maxwell equations are valid it this field theory if we identify the Noether current with the electromagnetic current.
Remember, that the homogeneous equations follow simply from

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{318}$$

$$A_\mu = (\Phi, -\vec{A})\,, \qquad \vec{B} = \vec{\nabla}\times\vec{A}, \qquad \vec{E} = -\dot{\vec{A}} - \vec{\nabla}\Phi \tag{319}$$

$$\vec{\nabla}\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}} \tag{320}$$

So if simply extend the global U(1) symmetry of the fermions to a global symmetry then we end up at quantum electrodynamics.
We could use other underlying symmetry groups as well. An example: **U(3) Symmetry**:
First, we enlange the configuration space by having three fermion fields in a bundle.

$$\psi(x) \to \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \tag{321}$$

This index $1 \ldots 3$ comes extra to the bispinor index or the chiral index.
A U(3) symmetry would mean that one can mix the three bispinors with a 3×3 unitary matrix:

$$\psi'(x) = U(x)\psi(x)\,. \tag{322}$$

If the Lagrangian is invariant to this transformation $\mathcal{L}[\psi'] = \mathcal{L}[\Psi]$ even if $U(x)$ is space-time dependent, then we have a U(3) gauge theory. For this to happen we must introduce gauge fields (A_μ) that transform with the bispinors.

For the fermion fields the transformation is simple, it commutes with γ^μ

$$\psi' = U\psi\,, \qquad \bar{\psi}' = \bar{\psi}U^+ \tag{323}$$

For $\mathcal{L}_D$:

$$\left[\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)\right]' = \bar{\psi}(x)U^+(x)i\gamma^\mu\partial_\mu U(x)\psi(x) - m\bar{\psi}U^+(x)U(x)\psi \tag{324}$$

The second term (mass) is invariant if U is unitary ($U^+U = \mathbf{1}$). To make the first term invariant we introduce the gauge fields:

$$D_\mu(x)\psi(x) = (\partial_\mu - igA_\mu(x))\psi(x) \tag{325}$$

$\mathcal{L}$ is invariant if we demand (analogously to electrodynamics):

$$(D_\mu(x)\psi(x))' = U(x)D_\mu(x)\psi(x) \tag{326}$$

$$\begin{aligned}
(\partial_\mu - igA_\mu)'\psi' &= U(\partial_\mu - igA_\mu)\psi \\
\partial_\mu - igA'_\mu &= U(\partial_\mu - igA_\mu)U^+ \\
A' &= UAU^+ + \frac{i}{g}[U\partial_\mu U^+ - \partial_\mu UU^+]
\end{aligned}$$

$$A' = UAU^+ - \frac{i}{g}[\partial_\mu, U]U^+ \tag{327}$$

A itself is a Hermitian 3×3 matrix.

We use the commutators to calculate the field strength tensor

$$[D_\mu, D_\nu] = igT^a F^a_{\mu\nu} \tag{328}$$

which, too, is a 3×3 Hermitian matrix. T^a Hermitian matrices are the generators of the symmetry

$$U(\theta) \approx 1 - iT^a\theta^a \tag{329}$$

We can also write

$$F_{\mu\nu} = F^a_{\mu\nu} T^a \qquad A_\mu = A^a_\mu T^a \qquad a = 1\ldots 8 \tag{330}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{331}$$

or, with the components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu \tag{332}$$

Since $D\psi$ transforms just as ψ, that goes over to $F_{\mu\nu}$, too.

$$[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = U(\theta)(D_\mu D_\nu - D_\nu D_\mu)\psi \tag{333}$$

So the transformatior rule for $F_{\mu\nu}$

$$F'_{\mu\nu} = U(\theta) F_{\mu\nu} U^+(\theta) \tag{334}$$

or, written in terms of the components of $F_{\mu\nu}$

$$F^a_{\mu\nu}{}' = F^a_{\mu\nu} + f^{abc}\theta^b F^c_{\mu\nu} \tag{335}$$

Now we have ingredients of the Lagrange function of Quantum Chromodynamics (QCD):

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \bar{\psi} i \gamma^\mu D_\mu \psi - m \bar{\psi} \psi \tag{336}$$

where F and D are defined as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{337}$$

$$D_\mu \psi = \left(\partial_\mu - i g T^a A^a_\mu \right) \psi \tag{338}$$

The dynamics of gauge fields is described by the term

$$\begin{aligned} -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} &= -\frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \\ &\quad - g f^{abc} \partial_\mu A^a_\nu A^{b,\mu} A^{c,\nu} \\ &\quad - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d,\mu} A^{e,\nu} \end{aligned} \tag{339}$$

Because of the non-abelian nature of the group the f^{abc} structure constants are not zero and, thus, there is self-interaction between the gauge fields.

In quantum chromodynamics the gauge particles are called *gluons*, the fermion fields are the quarks.

In electrodynamics the A vector potential was subject to a gauge transformation, but $F_{\mu\nu}$ was invariant.
In a non-abelian gauge theory the F field and the fermions transform at the same time, but not the same way:

$$\psi' = \psi - i\theta^a T^a \psi \quad (340)$$

$$F^a_{\mu\nu}{}' = F^a_{\mu\nu} + f^{abc}\theta^b F^c_{\mu\nu} \quad (341)$$

or for non-ifinitesimal transformations:

$$\psi' = U\psi \quad (342)$$

$$F'_{\mu\nu} = UF_{\mu\nu}U^+ \quad (343)$$

If in an $SU(N)$ gauge group the fermion fields have N components it is in the *fundamental representation*.
On the other hand the gauge fields are $N \times N$ matrices, they belong to the *adjoint* representation of the Lie Algebra.
It is possible to introduce matter fields that are not in the fundamental reperesntation (e.g. adjoint, or other). Nature has selected the fundamental representation for quarks in QCD.

Thus the simplest detectable particles are mesons ($\bar{q}q$) or baryons (qqq).
They are also representatios of the Lorentz group, and thus, mesons could have the spin 0,1, and the baryons can have $1/2$ or $3/2$.
The lightest hadron is the pion (a pseudoscalar with spin 0):

$$\pi^- = \bar{u}\gamma^5 d\,, \qquad \pi^+ = \bar{d}\gamma^5 u\,, \qquad \pi^0 = \frac{1}{2}\left(\bar{u}\gamma^5 u + \bar{d}\gamma^5 d\right) \tag{344}$$

these have a mass of about 135 MeV, which is much larger than the individual quark masses, or the difference between the quark masses.
If the quark masses are all equal then the Lagangian is invariant under the

$$q = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad q \to e^{-\frac{i}{2}\vec{\alpha}\vec{\tau}} q \tag{345}$$

isospin symmetry. Under this SU(2) symmetry group the three pions from a triplet.
If the quark masses are all zero then the Lagrangian is invariant under *chiral* rotations:

$$q \to e^{-\frac{i}{2}\vec{\beta}\vec{\tau}\gamma^5} q \tag{346}$$

The zero mass is required since the $\bar{q}q$ has no chiral symmetry. In Nature the spontaneous breaking of the chiral symmetry plays a more important role than the explicit breaking. The pions are the would-be Goldstone bosons. Their non-zero mass is due to the finite quark mass.

In total we have 6 quarks. These can be classified e.g. by electric charge and then we find two sorts:

Q=2/3	Q=-1/3
Up, 2.3 MeV	Down 4.8 MeV
Charm, 1275 MeV	Strange 95 MeV
Top, 173070 MeV	Bottom 4180 MeV

There is a baryon number $B = +1/3$ attached to each quark flavor. That naturally gives for mesons: $B = 1/3 - 1/3 = 0$ and for baryons $B = 1/3 + 1/3 + 1/3 = 1$. and for antibaryons $B = -1$.

Both the baryon number and the electric charge is conserved in QCD. From which symmetry do these follow?

$$Q: \qquad \begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} ue^{+i2/3\phi} \\ de^{-i1/3\phi} \end{pmatrix} \tag{347}$$

$$B: \qquad \begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} u \\ d \end{pmatrix} e^{i\theta/3} \tag{348}$$

The Q symmetry can actually be combined from the B and the 3rd component of the isospion (T_3).

Mesons	⟨Mass⟩	J^{PC}	I	S
π^-, π^0, π^+	138.0	0^{-+}	1	0
K^0, K^+	495.7	0^-	1/2	+1
$K^-, \overline{K}^0$				−1
η	547.3	0^{-+}	0	0
ρ^-, ρ^0, ρ^+	770.0	1^{--}	1	0
ω	781.9	1^{--}	0	0
$K^{\star 0}, K^{\star +}$	893.7	1^-	1/2	+1
$K^{\star -}, \overline{K}^{\star 0}$				−1
η'	957.8	0^{-+}	0	0
ϕ	1019.5	1^{--}	0	0

I: Isospin
S: Strangeness

J: Spin
P: parity
C: charge

Baryons	⟨Mass⟩	J^P	I	S
p, n	938.9	$1/2^+$	1/2	0
Λ	1116	$1/2^+$	0	-1
$\Sigma^-, \Sigma^0, \Sigma^+$	1193	$1/2^+$	1	-1
$\Delta^-, \Delta^0, \Delta^+, \Delta^{++}$	1232	$3/2^+$	3/2	0
Ξ^-, Ξ^0	1318	$1/2^+$	1/2	-2
$\Sigma^{\star -}, \Sigma^{\star 0}, \Sigma^{\star +}$	1385	$3/2^+$	1	-1
$\Xi^{\star -}, \Xi^{\star 0}$	1533	$3/2^+$	1/2	-2

Very small mass difference

(credit: Franz Muheim)

Isospin multiplets $|I, I_3\rangle$

$$\eta = |0,0\rangle$$

$$p = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \qquad n = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

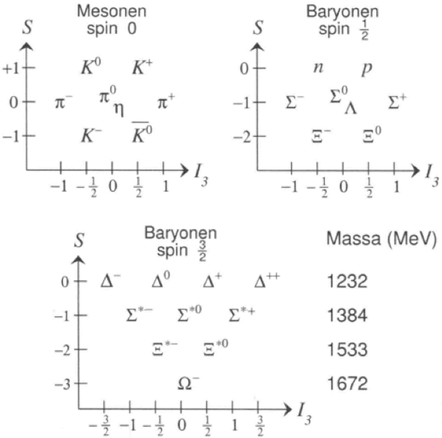
$$\pi^+ = |1,1\rangle \qquad \pi^0 = |1,0\rangle \qquad \pi^- = |1,-1\rangle$$

$$\Delta^{++} = \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle \qquad \Delta^{+} = \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle \qquad \Delta^{0} = \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle \qquad \Delta^{-} = \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle$$

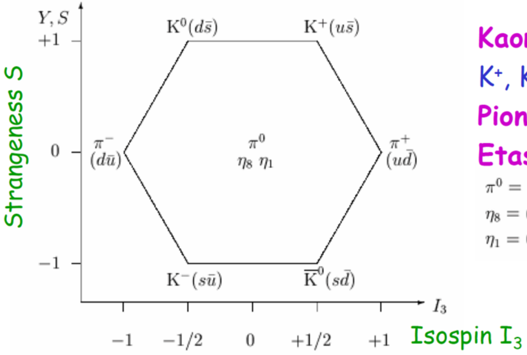
Conserved by the strong interactions:

Baryon number, strangeness, isospin, electric charge

$$Q = I_3 + \frac{1}{2}(S + B)$$


Mesonen
spin 0
S
+1
0
−1
K^0 K^+
π^- π^0 η π^+
K^- $\overline{K}^0$
−1 −½ 0 ½ 1
I_3
Baryonen
spin ½
S
0
−1
−2
n p
Σ^- Σ^0 Λ Σ^+
Ξ^- Ξ^0
−1 −½ 0 ½ 1
I_3
Baryonen
spin 3/2
S
0
−1
−2
−3
Δ^- Δ^0 Δ^+ Δ^{++}
Σ^{*-} Σ^{*0} Σ^{*+}
Ξ^{*-} Ξ^{*0}
Ω^-
−3/2 −1 −½ 0 ½ 1 3/2
I_3
Massa (MeV)
1232
1384
1533
1672

Pseudoscalar Mesons $J^P = 0^-$



Kaons:

K^+, K^0, anti-K^0, K^-

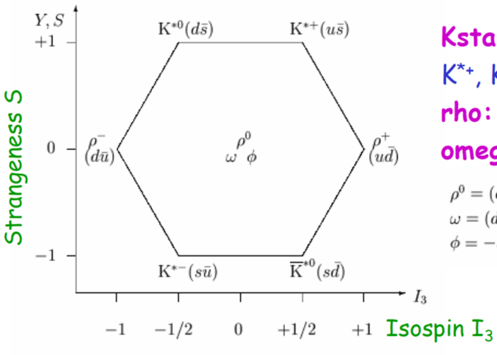
Pions: π^+, π^0, π^-

Etas: η, η'

$$\pi^0 = (d\bar{d} - u\bar{u})/\sqrt{2}$$
$$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$$
$$\eta_1 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$$

(credit: Franz Muheim)

Vector Mesons $J^P = 1^-$



Kstar:

K^{*+}, K^{*0}, anti-K^{*0}, K^{*-}

rho: ρ^+, ρ^0, ρ^-

omega/phi: ω, ϕ

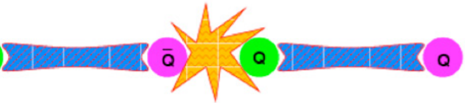
$$\rho^0 = (d\bar{d} - u\bar{u})/\sqrt{2}$$
$$\omega = (d\bar{d} + u\bar{u})/\sqrt{2}$$
$$\phi = -s\bar{s}$$

(credit: Franz Muheim)

The most important feature of QCD:

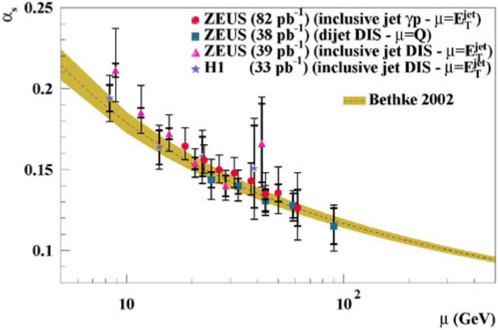
Asymptotic freedom

There is a coupling costant g is in the Lagrangian. It appears in $F_{\mu\nu}$ as the self-interaction of the gluons as well as in the covariant derivative. Even though this g is constant the effective coupling between quarks depends on their relative distance. The closer they are the less force they experience.



If a quark and antiquark are removed from one-anotheras vicinity, the attractive force is approximatively constant and energy is accumulated in the field. The energy is proportional to the distance: string. However, when this energy reaches the pion mass, the “spring” breaks and meson(s) are created.

Working in momentum space the small length scales correspond to large momenta and energies.



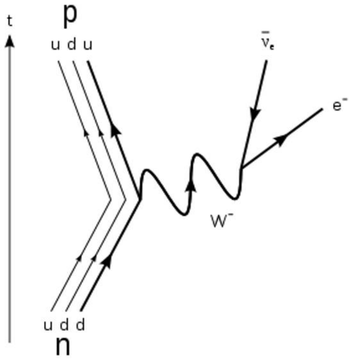
Asymptotic freedom in the HERA experiment

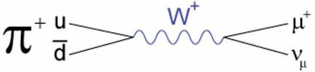
$$\alpha_s = \frac{g^2}{4\pi} \approx \frac{4\pi}{(11 - \frac{2}{3}n_f)\log \mu^2/\Lambda^2} \tag{349}$$

n_f: Number of quarks in the theory that are lighter than μ.
Λ: cca 200 MeV, a dimensionful coupling constant.

Typical processes are

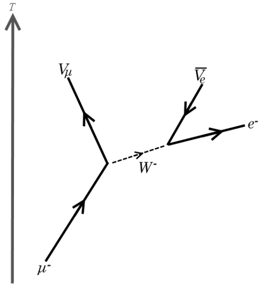
$$
\begin{aligned}
n \rightarrow p\, e\, \bar{\nu}_e \qquad & p\, \bar{\nu}_e \rightarrow e^+\, n\, \bar{\nu} \\
\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu \qquad & \mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu \\
\pi^+ \rightarrow \mu^+ + \nu_\mu \qquad & \pi^- \rightarrow \mu^- + \bar{\nu}_\mu
\end{aligned}
\tag{350}
$$





Feynman diagram for the muon decay:

The muon is an elementary particle similar to the electron Mass: 106 MeV, mean life time: $2.2 \cdot 10^{-6}$ s.



$$\mathcal{L}_a = -\frac{G_F}{\sqrt{2}} J_\lambda^+(x) J^\lambda + \text{h.c.} \tag{351}$$

$$J^\lambda(x) = \bar{\nu}_e \gamma^\lambda (1-\gamma^5) e + \bar{\nu}_\mu \gamma^\lambda (1-\gamma^5)\mu \tag{352}$$

h.c.: Hermitian conjugate, such that $\mathcal{L}$ becomes a real function.

$$\bar{\psi}\gamma^\lambda(1-\gamma^5)\psi = 2\bar{\psi}_L \gamma^\lambda \psi_L \tag{353}$$

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi \quad \psi_R = \tfrac{1}{2}(1+\gamma^5)\psi \quad \psi = \psi_L + \psi_R \tag{354}$$

This current describes only the *leptons* (electron, muon, tau and neutrinos):

$$J_l^\lambda(x) \quad = \quad \bar{\nu}_e\gamma^\lambda(1-\gamma^5)e + \bar{\nu}_\mu\gamma^\lambda(1-\gamma^5)\mu + \bar{\nu}_\tau\gamma^\lambda(1-\gamma^5)\tau \qquad (355)$$

There is one type of neutrino for the electron, muon and tau. In the standard model neutrinos are massless, but

$$\begin{aligned} m_e &= 0.000511 \text{ GeV} \\ m_\mu &= 0.1066 \text{ GeV} \\ m_\tau &= 0.777 \text{ GeV} \end{aligned} \qquad (356)$$

Notice that the right-handed leptons are not part of J_l.
Let us include the hadrons, too:

$$J^\lambda(x) = J_h^\lambda(x) + J_l^\lambda(x) \qquad (357)$$

Considering only u and d quarks one could write:

$$J_h^\lambda(x) = \bar{u}\gamma^\lambda(1-\gamma^5)d \qquad (358)$$

This current can account for the β-decay as well as for the pion decay.

Hadrons with strange quarks, too, show similar decay modes:

Leptonic decay of strange hadrons:

$$
\begin{array}{lll}
\underbrace{K^+}_{u\bar{s}} \to \mu^+\nu & \underbrace{K^+}_{u\bar{s}} \to \mu^+\nu \underbrace{\pi^0}_{u\bar{u}+d\bar{d}} & \underbrace{K^0}_{d\bar{s}} \to \mu^+\nu \underbrace{\pi^-}_{d\bar{u}} \\
\underbrace{\Lambda}_{uds} \to \underbrace{p}_{uud} \ell^-\bar{\nu} & \underbrace{\Sigma^-}_{dds} \to \underbrace{n}_{udd} \ell^-\bar{\nu} & \underbrace{\Xi^0}_{uds} \to \underbrace{\Sigma^+}_{uus} \ell^-\bar{\nu} \\
\underbrace{\Xi^-}_{dss} \to \underbrace{\Sigma^0}_{uud} \ell^-\bar{\nu} & \underbrace{\Xi^-}_{dss} \to \underbrace{\Lambda}_{uds} \ell^-\bar{\nu} & \underbrace{\Omega^-}_{sss} \to \underbrace{\Xi^0}_{uss} \ell^-\bar{\nu}
\end{array} \tag{359}
$$

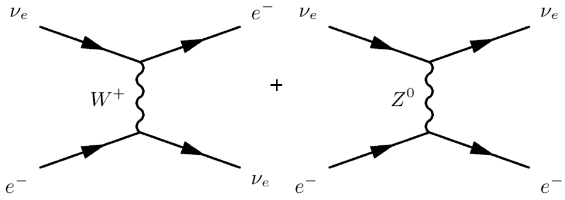
ℓ^- is a lepton with negative electric charge. The neutrino must also belong to the same family.

In all cases:

- $|\Delta S| \leq 1$, only one of the strange quarks decays
- $\Delta Q = \Delta S$, i.e. there are $s \to u$ and $\bar{s} \to \bar{u}$ processes, but no $s \to d$.

$$
\begin{aligned}
J_h^\lambda(x) &= c_1 \bar{u}\gamma^\lambda(1-\gamma^5)d + c_2\bar{u}\gamma^\lambda(1-\gamma^5)s \\
&= \bar{u}\gamma^\lambda(1-\gamma^5)d\cos\theta_c + \bar{u}\gamma^\lambda(1-\gamma^5)s\sin\theta_c && (360) \\
J_h^\lambda(x) &= \bar{u}\gamma^\lambda(1-\gamma^5)d_\theta && (361) \\
d_\theta &= \cos\theta_c d + \sin\theta_c s && (362)
\end{aligned}
$$

Not only the W boson (80 GeV) can mediate weak interactions. The Z boson (91 GeV).



The Z and the photon have vary similar effect: intercation without flavor changing. But: The Z boson can connect to neutral particles, too, like the neutrino. The coupling constants to the Z boson reveal a structure:

$$g_L^\nu = \frac{1}{2} \qquad g_R^\nu = 0 \tag{363}$$

$$g_L^e = -\frac{1}{2} + \sin^2\theta_W \qquad g_R^e = \sin^2\theta_W \tag{364}$$

$$g_L^u = \frac{1}{2} - \frac{2}{3}\sin^2\theta_W \qquad g_R^u = -\frac{2}{3}\sin^2\theta_W \tag{365}$$

$$g_L^d = -\frac{1}{2} + \frac{1}{3}\sin^2\theta_W \qquad g_R^d = \frac{1}{3}\sin^2\theta_W \tag{366}$$

where the Weinberg angle is introduced. This angle describes the relation between the Z boson and the photon in the Standard Model.

The four fermion action gives a reasonable description of the weak decays with $G_f \approx 10^{-5}$ GeV

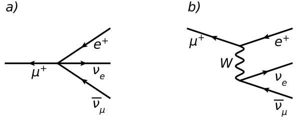
It has a shortcoming, thought, the quantum theory is ill-defined on a continuous space-time (*not perturbatively renormalizable*).

In general theories where the coupling constant has a negative mass dimensions are not renormalizable.

If we had a W boson, the four-fermion action can be split up to

$$\mathcal{L} = gJ_\mu W^\mu + \text{h.c.} \tag{367}$$

Here we use the gauge fields that got mass through the Higgs mechanism. What used to be a point interaction appears no in the second order of perturbation theory using the Feynman diagram:



The wavy line is the propagator of W. On the vertices one writes the coefficients in the Lagrangian, for the propagators we write:

$$\Delta_{\mu\nu} = -\frac{g^{\mu\nu} - k_\mu k_\nu / M_W^2}{k^2 - M_W^2 + i\epsilon} \tag{368}$$

In the low energy limit ($k \ll M_W$): $G_F/\sqrt{2} = g^2/8M_W^2$

Gauge symmetry can be formulated both for fermions and scalars. Let's have for example a scalar field with several components and a local symmetry:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + (D_{ij}\Phi_j)^+ (D_{ik}\Phi_k) - V(\Phi) \tag{369}$$

Φ is in some representation *(e.g. fundamental)* of the gauge group *(e.g. SU(N))*, then $j = 1..N$. Of course, V must also be symmetric, it is sufficient if V is a funciton of $\sum_j \phi_j^+ \phi_j$.
If V happens to have its minimum away from the origin $\phi_j \equiv 0$, then it will have a manifold of minima. Let's pick one: $\bar{\Phi}_j$. Then we can intoduce new variables $(\phi_j(x))$:

$$\Phi_j(x) = \bar{\Phi}_j(x) + \phi_j(x) \tag{370}$$

and expand V aronud $\bar{\Phi}_j$.

$$V(\phi + \bar{\Phi}) = \sum_{ij} \phi_i M_{ij} \phi_j + \Delta V(\phi) \tag{371}$$

There is also a constant term, which is just a shift in the energy.
The linear term is zero by construction: V has a minimum at $\bar{\Phi}$

We can diagonalize the mass matrix M_{ij} in

$$V(\phi + \bar{\Phi}) = \sum_{ij} \phi_i M_{ij} \phi_j + \Delta V(\phi) \tag{372}$$

as $M = U^+ M_{\mathrm{diag}} U$ and switch to new variables: $\varphi = U\phi$.

$$\begin{aligned} (D\phi)^+(D\phi) &= (D\phi)^+ U^+ U (D\phi) = (DU\phi)^+(D\phi) = (D\varphi)^+(D\varphi) && (373) \\ \phi^+ M \phi &= \phi^+ U^+ M_{\mathrm{diag}} U \phi = (U\phi)^+ M U \phi = \varphi^+ M_{\mathrm{diag}} \varphi && (374) \end{aligned}$$

The eigenvalues of M are the new masses $M_{\mathrm{diag}} = \mathrm{diag}(m_1^2, m_2^2, \ldots)$:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \sum_j \left[(D\varphi_j)^+(D\varphi_j) - (\varphi_i)^+ m_j^2 \varphi_j \right] - \Delta V + \ldots \tag{375}$$

The eigenvalues m_j^2 are actually the eigenvalues of the derivative matrix of the potential at its minimum.

$$M = \begin{pmatrix} \frac{\partial^2 V}{\partial\phi_1 \partial\phi_1} & \frac{\partial^2 V}{\partial\phi_1 \partial\phi_2} & \cdots \\ \frac{\partial^2 V}{\partial\phi_2 \partial\phi_1} & \frac{\partial^2 V}{\partial\phi_2 \partial\phi_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{376}$$

We have already discussed both local and global symmetries, with both abelian and non-abelian underlying groups.
We have seen that if the otherwise symmetric potential's minimum is at field values which are not symmetric (non-symmetric ground state), then massless states emerge through the Goldstone theorem.
Spontaneous symmetry breaking (SSB)
A generalization: *What happens if a local gauge symmetry meets a symmetry breaking potential?*
Let's take the example of the SU(2) symmetry group and add a scalar field with two components (fundamental representation):

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{377}$$

The following Lagrangian is then gauge invariant under local SU(2) transformations:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu \phi)^+ (D^\mu \phi) - V(\phi) && (378)\\
D_\mu \phi &= (\partial_\mu - igT^a A^a)\,\phi \qquad T^a = \sigma^a/2 && (379)\\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu && (380)\\
V(\phi) &= -\mu^2 (\phi^+ \phi) + \lambda (\phi^+ \phi)^2 && (381)
\end{aligned}$$

If $\mu^2 > 0$ then its minimum is not in the origin, but

$$\phi^+\phi\big|_{\text{ground state}} = v^2/2\,, \qquad \text{with} \;\; v = \sqrt{\mu^2/\lambda} \tag{382}$$

A generic minimum location can be rotated into this ground state (with the same energy):

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}\,. \tag{383}$$

We again study the displaced field's fluctuations:

$$\phi' = \phi - \phi_0 \tag{384}$$

In the term with the covariant derivative of the scalar field:

$$(D_\mu\phi)^+(D^\mu\phi) \quad = \quad \left(\left(\partial_\mu - igT^aA^a_\mu\right)(\phi' + \phi_0)\right)^+\left(\left(\partial^\mu - igT^aA^{a\mu}\right)(\phi' + \phi_0)\right)$$

we get a term

$$g^2\phi_0^+T^aT^b\phi_0A^a_\mu A^{b\mu} = \frac{1}{2}\left(\frac{gv}{2}\right)^2 A^a_\mu A^{a\mu} \tag{385}$$

The combination $(D_\mu\phi)^+(D^\mu\phi)$ gives rise to the following interaction terms

$$\phi'\phi' AA \quad \phi' AA \quad \phi'(\partial\phi')A \tag{386}$$

Every term with three or more fields is an interaction term, from the quadratic terms emerge the masses and there is no linear term, if we expand around the ground state.

mass generated for the gauge boson: $$\frac{1}{2}\left(\frac{gv}{2}\right)^2 = \frac{1}{2}M_A^2 A_\mu A^\mu \tag{387}$$

$M_A = gv/2$. Such a term is clearly not gauge invariant. Yet $\mathcal{L}$ is still gauge invariant, since the ground state ϕ_0 does transform under the gauge group. But whichever ground state we rotate into, the gauge fileds will always experience the same effective mass term $\sim A_\mu A^\mu$.

In the scalar sector of the model:

$$\begin{aligned} \phi^+\phi &= \phi'^+\phi' + \phi_0^+\phi' + \phi'^+\phi_0 + \phi_0^+\phi_0 \\ (\phi^+\phi)^2 &= v^2\phi' + (\phi_0^+\phi' + \phi'^+\phi_0)^2 + \ldots \end{aligned} \tag{388}$$

The masses are the coefficients of the quadratic term in ϕ':

$$\frac{\lambda v^2}{2}(\phi_2' + \phi'^+_2)^2 = \frac{\mu^2}{2}(\phi_2' + \phi'^+_2)^2 + 0\cdot(\phi_2' - \phi'^+_2)^2 \tag{389}$$

We again find a massless combinaton e.g. $\phi_2' - \phi_2'^+$, these are the Goldstone modes. We call the other combination with mositive mass $(\phi_2' + \phi_2'^+)$ the *Higgs field*.

To better identify the degrees of freedom, let's switch to a special gauge fixing, the *unitary gauge*.

In every space-time point we choose $\xi^a(x)$ such that

$$\phi(x) = \exp\left\{\frac{2i}{v} T^a \xi^a(x)\right\} \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix}, \tag{390}$$

and $\eta \equiv 0$ and $\xi^a \equiv 0$ describe the gronud state. We can redefine the fields:

$$\begin{aligned} \Phi(x) &= U(x)\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix} && (391) \\ T^a B^a &= U(x) T^a A_\mu^a(x) U^{-1}(x) - \frac{i}{g} \left[\partial_\mu U(x)\right] U^{-1}(x) && (392) \end{aligned}$$

with

$$U(x) = \exp\left\{-\frac{2i}{v} T^a \xi^a(x)\right\} \tag{393}$$

From the gauge transformation rules follow:

$$\begin{aligned} D_\mu\phi &= U^{-1}(x) D_\mu \Phi && (394)\\ F^a_{\mu\nu}F^{a\mu\nu} &= G^a_{\mu\nu}G^{a\mu\nu} && (395)\end{aligned}$$

with

$$\begin{aligned} D_\mu\Phi &= (\partial_\mu - igTB_\mu)\,\Phi && (396)\\ G^a_{\mu\nu} &= (\partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g\epsilon^{abc}B^b_\mu B^c_\nu)\,. && (397)\end{aligned}$$

Now we can write the Lagrangian in terms of G and Φ:

$$\mathcal{L} = (D_\mu\Phi^\mu)^+(D^\mu\Phi) + \frac{\mu^2}{2}(v+\eta)^2 - \frac{\lambda}{4}(v+\eta)^4 - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} \qquad (398)$$

From the first term we can extract the part quadratic in B:

$$\frac{g^2}{2}(0,v)T^aT^b\begin{pmatrix}0\\ v\end{pmatrix}B^a_\mu B^{b\mu} = \frac{1}{2}\left(\frac{gv}{2}\right)^2 B_\mu B^\mu \qquad (399)$$

From the two complex scalar field components we are left with one real field. $\xi(x)$ played the role of a gauge transformation, it does not appear in the new form of the Lagrangian. The three gauge fields become massive.

So far we worked with classical fields that are simple functions of space and time.
In quantum mechanics we use operators to describe the coordinates of point particles.

$$x_i \quad \to \quad \hat{x}_i, \qquad i = 1 \ldots 3 \tag{400}$$

and similarly for the momentum:

$$p_i \quad \to \quad \hat{p}_i, \qquad i = 1 \ldots 3 \tag{401}$$

The operators are infinite matrices in an abstract Hilbert space. As usual with matrices they do not commute, or if they do, that has special consequences.

$$[\hat{x}_i, \hat{x}_j] \quad = \quad 0 \tag{402}$$

$$[\hat{p}_i, \hat{p}_j] \quad = \quad 0 \tag{403}$$

$$[\hat{p}_i, \hat{x}_j] \quad = \quad \frac{\hbar}{i} \delta_{ij} \tag{404}$$

In field theory, just like in mechanics, we seek the canonical coordinates and momenta. At every space point there is a small oscillator with the canonical coordinate $\Phi(\vec{x}, t)$, where $\vec{x}$ is just a label for the coupled oscillator.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 \tag{405}$$

The definition of the canonical momentum Π reads just like in mechanics:

$$\Pi = \frac{\delta\mathcal{L}}{\delta\partial_0\Phi} \tag{406}$$

or, in our case:

$$\Pi(x,t) = \partial^0\Phi = \frac{d\Phi(x,t)}{dt} \tag{407}$$

For a fixed time (t), this momentum commutes with all operators that are not defined at the same position $\vec{x}$.

$$[\hat{\Phi}_i(\vec{x}, t), \hat{\Phi}_j(\vec{y}, t)] = 0 \tag{408}$$

$$[\hat{\Pi}_i(\vec{x}, t), \hat{\Pi}_j(\vec{y}, t)] = 0 \tag{409}$$

$$[\hat{\Pi}_i(\vec{x}, t), \hat{\Phi}_j(\vec{y}, t)] = -i\delta(\vec{x} - \vec{y}) \tag{410}$$

A special role is played by the Hamilton operator, which determines the time evolution.

$$\begin{aligned} \partial_0 \hat{\Phi}(\vec{x}, t) &= i[\hat{H}, \hat{\Phi}(\vec{x}, t)] \\ \partial_0 \hat{\Pi}(\vec{x}, t) &= i[\hat{H}, \hat{\Pi}(\vec{x}, t)] \,. \end{aligned} \tag{411}$$

As in mechanics, the Hamilton oporator is derived from the Hamilton function:

$$H(\Phi, \Pi) = \int d^3x \, [\Pi \partial_0 \Phi - \mathcal{L}(\Phi, \Pi)] \tag{412}$$

$$\hat{H} = H(\hat{\Phi}, \hat{\Pi}) = \int d^3x \frac{1}{2} \left[\Pi^2(\vec{x}, t) + (\nabla \Phi(\vec{x}, t))^2 + m^2 \Phi^2(\vec{x}, t) \right] \tag{413}$$

In this simple case (without interaction) the equations of motion is

$$\begin{aligned} \partial_0 \hat{\Phi}(\vec{x}, t) &= \hat{\Pi}(\vec{x}, t) \\ \partial_0 \hat{\Pi}(\vec{x}, t) &= \triangle \hat{\Phi}(\vec{x}, t) - m^2 \hat{\Phi}(\vec{x}, t) \end{aligned} \tag{414}$$

i.e. the Klein-Gordon equation is satisfied on the operator level.

$$(\partial^2 + m^2) \hat{\Phi}(\vec{x}, t) = 0 \tag{415}$$

This can be generalized to the interaction case, too (e.g. $\mathcal{L}_i = \lambda \Phi^4$)

If we have a set of oscillators, a linear combination of these oscillators could equally be used to define the system. Writing the operators in Fourier space we find that the oscillators decouple. We can indtroduce the ladder operators (raising and lowering operators):

$$\hat{\Phi}_S(\vec{x}) = \int \frac{d^3k}{[(2\pi)^3]^{1/2}} \sqrt{\frac{1}{2\omega_k}} \left[\hat{a}_S(\vec{k})e^{i\vec{k}\vec{x}} + \hat{a}_S^+(\vec{k})e^{-i\vec{k}\vec{x}}\right] \tag{416}$$

$$\hat{\Pi}_S(\vec{x}) = -i \int \frac{d^3k}{[(2\pi)^3]^{1/2}} \sqrt{\frac{\omega_k}{2}} \left[\hat{a}_S(\vec{k})e^{i\vec{k}\vec{x}} - \hat{a}_S^+(\vec{k})e^{-i\vec{k}\vec{x}}\right] \tag{417}$$

We use the notation $\omega_k = \sqrt{m^2 + |\vec{k}|^2}$. The relation turned around:

$$\hat{a}_S(k) = \int \frac{d^3k}{[(2\pi)^3 2\omega_k]^{1/2}} \left[\omega_k \hat{\Phi}_S(\vec{x}) - i\hat{\Pi}_S(\vec{x})\right] e^{i\vec{k}\vec{x}} \tag{418}$$

$$\hat{a}_S^+(k) = \int \frac{d^3k}{[(2\pi)^3 2\omega_k]^{1/2}} \left[\omega_k \hat{\Phi}_S(\vec{x}) + i\hat{\Pi}_S(\vec{x})\right] e^{-i\vec{k}\vec{x}} \tag{419}$$

And so we get the "diagonalized" form of the Hamilton operator for a free field theory.

$$\hat{H} = \int d^3 k \omega_k \hat{a}_S^+(k) \hat{a}_S(k) \tag{420}$$

Working out the commutation relations for $\hat{a}_S$ and $\hat{a}_S^+$ (these are the same as in Heisenberg picture):

$$[\hat{a}(k), \hat{a}(k')] = 0 \tag{421}$$

$$[\hat{a}^+(k), \hat{a}^+(k')] = 0 \tag{422}$$

$$[\hat{a}(k), \hat{a}^+(k')] = \delta(k - k') \tag{423}$$

Remember, that in Heisenberg picture $\hat{a}(t)$ rotates with $e^{i\omega t}$:

$$\hat{a}_H(k, t) = \int \frac{d^3 k}{[(2\pi)^3 2\omega_k]^{1/2}} \left[\omega_k \hat{\Phi}_S(\vec{x}) - i\hat{\Pi}_S(\vec{x})\right] e^{i\vec{k}\vec{x} - i\omega_k t} \tag{424}$$

This makes for the field operator:

$$\hat{\Phi}(\vec{x}, t) = \int \frac{d^3 k}{[(2\pi)^3 2\omega_k]^{1/2}} \left[\hat{a}_S(\vec{k}) e^{i(\vec{k}\vec{x} - \omega_k t)} + \hat{a}_S^+(\vec{k}) e^{-i(\vec{k}\vec{x} - \omega_k t)}\right] \tag{425}$$

The free scalar field theory is, thus, equivalent to a set of independent quantum oscillators, one for each spatial momentum.

How shall we imagine, what the Hilbert space is like?
For one harmonic oscillator the wave function is a vector in some Hilbert space, with the ground state energy ($E_0 = \frac{1}{2}\hbar\omega$).
We do not want to write down these wave functions explicitly, it is enough to know that they exist, and can be generated by the raising operator from the ground state $|0\rangle$: The first and the n-th excited state with $E_n = \hbar\omega(\frac{1}{2} + n)$ can be written as

$$|1\rangle = \hat{a}^+|0\rangle \qquad |n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle \tag{426}$$

The one particle state we can written as the first excitation of the vacuum. If the particle has a momentum $\vec{p}$ the the oscillator labbeled by $\vec{p}$ is excited by one raising operator, starting from the vacuum ($|0\rangle$):

$$|\vec{k}\rangle = [(2\pi)^3 2\omega_k]^{1/2} a^+(\vec{k})|0\rangle \tag{427}$$

The entire space of this basis of one-, two-, etc states span the *Fock space*: the *direct product* of the Hilbert spaces of the individual oscillators.
Bosonic particles are defined as the first excited states of the corresponding oscillators in the configuration space of the field.
It is possible to have several particles with the same momentum (Bose-Einstein Condensate), it corresponds to a higher excitation of the same oscillator.

For fermions the Pauli principle forbids the simultaneous existence of particles in the same state. Thus, we cannot define fermionic particles as oscillator states. Actually the Pauli principle requires in quantum mechanics that the wave function is anti-symmetric for the exchange of two degrees of freedom. For massive fermions (e.g. electrons) we had this Lagrangian:

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{428}$$

the canonical momentum can be easily obtained:

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi(x)} = i\psi^+(x) \tag{429}$$

in order to get the Pauli principle we postulate not commutation but anti-commutation relations:

$$\begin{aligned} \{\psi(\vec{x},t), \vec{\psi}(\vec{y},t)\} &= 0 \\ \{\psi^+(\vec{x},t), \vec{\psi}^+(\vec{y},t)\} &= 0 \\ \{\psi(\vec{x},t), \vec{\psi}^+(\vec{y},t)\} &= \delta^3(\vec{x}-\vec{y}) \end{aligned} \tag{430}$$

In the classical limit $\{\psi(\vec{x},t), \vec{\psi}^+(\vec{y},t)\} = 0$, thus all spinor fields are anti-commutating.

Even with $\hbar \to 0$ they are not normal numbers: Grassmann variables

Analogously to the bosons we can rewrite the field operator in momentum space and use ladder operators:

$$\hat{\psi} = \int \frac{d^3p}{[(2\pi)^3 2\omega_p]^{1/2}} \left[u_s(\vec{k})\hat{a}_s(\vec{k})e^{i(\vec{k}\vec{x}-\omega_k t)} + v_s(\vec{k})\hat{b}_s^{+}(\vec{k})e^{-i(\vec{k}\vec{x}-\omega_k t)} \right] \quad (431)$$

$$\hat{\bar{\psi}} = \int \frac{d^3p}{[(2\pi)^3 2\omega_p]^{1/2}} \left[\bar{v}_s(\vec{k})\hat{b}_s(\vec{k})e^{i(\vec{k}\vec{x}-\omega_k t)} + \bar{u}_s(\vec{k})\hat{a}_s^{+}(\vec{k})e^{-i(\vec{k}\vec{x}-\omega_k t)} \right] \quad (432)$$

$$\{\hat{a}_i(\vec{p}), \hat{a}_j^{+}(\vec{q})\} = \{\hat{b}_i(\vec{p}), \hat{b}_j^{+}(\vec{q})\} = (2\pi)^3\delta(\vec{p}-\vec{q})\delta_{ij} \quad (433)$$

$$\{\hat{a}_i(\vec{p}), \hat{a}_j(\vec{q})\} = \{\hat{b}_i(\vec{p}), \hat{b}_j(\vec{q})\} = 0 \quad (434)$$

$$\{\hat{a}_i^{+}(\vec{p}), \hat{a}_j^{+}(\vec{q})\} = \{\hat{b}_i^{+}(\vec{p}), \hat{b}_j^{+}(\vec{q})\} = 0 \quad (435)$$

$$\{\hat{a}_i(\vec{p}), \hat{b}_j(\vec{q})\} = \{\hat{a}_i(\vec{p}), \hat{b}_j^{+}(\vec{q})\} = 0 \quad (436)$$

Then the Hamilton operator will take the form

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3}\omega_p \left(\hat{a}_s^{+}(\vec{p})\hat{a}_s(\vec{p}) + \hat{b}_s^{+}(\vec{p})\hat{b}_s(\vec{p}) \right) \quad (437)$$

The anti-commutation rules have the immediate consequence that the double excitation (or annihilation) operator is zero.

$$
\begin{aligned}
[\hat{a}^+(\vec{p})]^2 &= [\hat{a}(\vec{p})]^2 = 0 && (438) \\
[\hat{b}^+(\vec{p})]^2 &= [\hat{b}(\vec{p})]^2 = 0 && (439)
\end{aligned}
$$

Starting from the vacuum $|0\rangle$, which is the ground state of the Fock space:

$$
\begin{aligned}
a_s^+(\vec{p})|0\rangle & & \text{One fermion state} \\
[a_s^+(\vec{p})]^2|0\rangle &= 0 & \text{This state does not exist (Pauli principle)} \\
a_{s_1}^+(\vec{p}_1)a_{s_2}^+(\vec{p}_2)|0\rangle & & \text{Two particles in different state} \\
b_s^+(\vec{p})|0\rangle & & \text{One anti-particle} \\
a_{s_1}^+(\vec{p}_1)b_{s_2}^+(\vec{p}_2)|0\rangle & & \text{partcile and anti-particle} && (440)
\end{aligned}
$$

The Fock space for fermions has quantum numbers 0 and 1 (for both spinor components, but also for both particle and anti-particle). For every classical degree of freedom there is a two-state system. The Fock space is a direct product space.

The two-point function is a quantum average of the product of two fields:

$$G^>(x,y) = \left\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\right\rangle = G^<(y,x) \tag{441}$$

In the free theory this can be easily calculated:

$$G^>(x,y) = \left\langle 0| \int \frac{d^3k}{((2\pi)^3 2\omega_k)^{1/2}} \int \frac{d^3p}{((2\pi)^3 2\omega_p)^{1/2}} \left(\hat{a}_k e^{-ikx} + \hat{a}_k^+ e^{ikx}\right) \left(\hat{a}_p e^{-ipy} + \hat{a}_p^+ e^{ipy}\right) |0 \tag{442}$$

Keeping in mind that $\langle 0|a_p^+ = 0$ and $a|0\rangle = 0$ and having to commutators

$$[\hat{a}(k), \hat{a}^+(k')] = \delta(k-k') \tag{443}$$

$$G^>(x,y) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-ip(x-y)} = G^>(x-y) \tag{444}$$

Several other forms of the two-point function exists:

$$\rho(x,y) = i(G^>(x,y) - G^<(x,y)) \quad \text{spectral function} \tag{445}$$

$$F(x,y) = \frac{1}{2}(G^>(x,y) + G^<(x,y)) \quad \text{symmetric propagator} \tag{446}$$

$$G^R(x,y) = \theta(x_0 - y_0)\rho(x,y) \quad \text{retarded propagator} \tag{447}$$

$$G^A(x,y) = -\theta(y_0 - x_0)\rho(x,y) \quad \text{advanced propagator} \tag{448}$$

One particularly useful propagator is the time ordered propagator:

$$G(x,y) = \left\langle 0 | T\hat{\phi}(x)\hat{\phi}(y) | 0 \right\rangle = \theta(x_0 - y_0) G^{>}(x,y) + \theta(y_0 - x_0) G^{<}(x,y) \quad (449)$$

Using the commutation relation of a and a^{+} we have

$$G(x,y) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}(\vec{x}-\vec{y}) - i\omega|x_0 - y_0|}}{2\omega_p} \quad (450)$$

Statement: This propagator can also be written as

$$G(x,y) = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} \quad (451)$$

where

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ikx} \quad (452)$$

is called the Feynman propagator.

The free propagator solves the equation of motion of the free theory:

$$(\partial^2 + m^2)G^>(x,y) = 0 \tag{453}$$

Whereas for the Feynman propagator:

$$(\partial^2 + m^2)G(x,y) = -i\delta(x-y) \tag{454}$$

To show this we used the commutation relation:

$$\delta(x_0 - y_0)\frac{\partial}{\partial x_0}\rho(x,y) = \delta(x_0 - y_0) \tag{455}$$

Using the identity

$$\frac{1}{\alpha \pm i\epsilon} = \mp i\pi\delta(\alpha) + \mathcal{P}\frac{1}{\alpha} \tag{456}$$

we can write in Fourier space

$$G_R(\omega) = \int \frac{d\omega'}{2\pi}\frac{i\rho(\omega')}{\omega - \omega' + i\epsilon} \tag{457}$$

$$2\mathrm{Im}\ G_R(\omega) = \mathrm{Im}\ \rho(\omega) \tag{458}$$

Equations of motion for the interacting case

$$(\partial_x^2 + m^2)G_{ij}^>(x,y) = \left\langle \frac{\delta\mathcal{L}_i}{\delta\varphi_i(x)}\varphi_j(y)\right\rangle \tag{459}$$

$$(\partial_x^2 + m^2)G_{ij}(x,y) = \left\langle \mathcal{T}\frac{\delta\mathcal{L}_i}{\delta\varphi_i(x)}\varphi_j(y)\right\rangle - i\delta_c(x,y) \tag{460}$$

There are two important elementary processes that appear in a typical setting in quantum field theory:

- $1 \to n$ Decay: one particle emits n decay products
- $2 \to n$ Scattering: collision of two incoming particles, n particles leave the scene.
  Elastic scattering: the same two particles fly on (after changing their direction)
  Inelastic scattering: a reaction occurs, different (possibly more) particles are created

In both cases there was a transition between the initial and final quantum states. In the *Fock space* both the initial and final states appear as single or multi-particle states (at least approximately).
Perturbation theory calculates the probability of such transitions. *Feynman diagrams* are the graphical representation of the corresponding formulas in perturbation theory. The order of perturbative expansion is given by the number of vertices in the used set of Feynman diagrams.

$\lvert\psi_i\rangle = \lvert\psi(t=-\infty)\rangle$	Initial state
$\lvert\psi_f\rangle$	One of the final state to which a reaction might occur, all orthogonal
$i\frac{d}{dt}\lvert\psi\rangle = \hat{H}\lvert\psi\rangle$	Schrödinger's equation
$\lvert\psi(t)\rangle = e^{-i\int_{-\infty}^{t}\hat{H}d\tau}\lvert\psi_i\rangle$	Generic solution to Schrödinger's equation
$\lvert\psi(t=\infty)\rangle = \alpha\lvert i\rangle + \sum_f \beta_f\lvert\psi_f\rangle$	The result at $t=\infty$
$\lvert\beta_f\rvert^2$	Probability for the final state f
$\beta_f = \langle\psi_f\lvert\psi(t=\infty)\rangle$	One of the possible final states selected
$\beta_f = T_{fi} = \langle f\lvert e^{-i\int_{-\infty}^{\infty}Hdt}\rvert i\rangle$	Transition amplitude
$T_{fi} = (2\pi)^4\delta^4(p_f - p_i)M_{fi}$	4-momentum can be factorized

Let's calculate the probability of the actual transition into one of the possible final states:

$$\bar{w}_{fi} = |T_{fi}|^2 = (2\pi)^4 \delta^4(p_f - p_i)(2\pi)^4 \delta^4(p_f - p_i)|M|^2 \tag{461}$$

The product of two δ functions will be problematic. However, we know, that the experiment takes place in a finite range of space and time. In finite volume the momenta are discretized, e.g.

$$p_x = j\frac{2\pi}{L}, \qquad j = -\infty \ldots \infty \tag{462}$$

The relation of the continuum and discretized momentum integral of an arbitrary function f in one dimension:

$$\int \frac{dp}{2\pi} f(p) \to \sum_{j=-\infty}^{\infty} \frac{\Delta p}{2\pi} f(p) = \frac{1}{L} \sum_{j=-\infty}^{\infty} f\left(j\frac{2\pi}{L}\right) \tag{463}$$

$$\int \frac{dp}{2\pi} \delta(p) = \frac{1}{2\pi}\,, \quad \text{thus} \quad \frac{1}{L} \sum_{j=-\infty}^{\infty} (L\delta_{j0}) = \frac{1}{2\pi}\,, \quad \text{hence} \quad \delta(0) \to \frac{L}{2\pi} \tag{464}$$

In 4D within a spatial volume (V) and time frame (T):

$$(2\pi)^4 \delta^4(0) = VT\,, \tag{465}$$

There are, of course, many final states possible, e.g. the decay products may fly off in many possible directions with various momenta. Each final state has an infinitesimal cell in the phase space.
If we have an n-particle state, the phase volume reads:

$$d\bar{\Phi} = \Pi_{l=1}^{n} \frac{d\vec{k}_l V}{(2\pi)^3} \tag{466}$$

The number of particles in the initial state are normalized such that there are $2\omega_k$ particles in one unit volume.

$$N = \Pi_{l=1}^{n}(2\omega_{k_l} V) \tag{467}$$

For a two-particle initial state (scattering) with $\vec{p}_A$ and $\vec{p}_B$:

$$N = (2\omega_{p_B} V)(\omega_{p_A} V) \tag{468}$$

Thus, we can write the transition probability into an infinitesimal cell of the phase space for a unit time:

$$dw_{fi} = \frac{\bar{w}_{fi}}{T}\frac{d\bar{\Phi}}{N} = \frac{V|M|^2}{\Pi_{i=1}^{k}(2E_i V)} d\Phi \tag{469}$$

with

$$d\Phi = (2\pi)^4 \delta^4(p_f - p_i) \Pi_{l=1}^{n} \frac{d\vec{k}_l}{(2\pi)^3 2E_l} \tag{470}$$

In the special case of $n = 1$ a decay process is described. We can calculate a *decay rate*. For a well defined final state:

$$d\Gamma = \frac{1}{2E_a}|M|^2 d\Phi \tag{471}$$

Since a decay is possible into many states, all the possible final states have to be considered and integrated:

$$\Gamma = \int d\Gamma \tag{472}$$

This integral also means that all spin states need to be summed up.
How about the spin state (polarization) of the incoming particle? It is usually not know, thus, we will average over the possible initial spin states.

If there are two particles in the initial state, their (mutual) *cross section* σ can be given.
Let's imagine a particle a sitting in Lab frame. It is hit by a beam of particles b.
If there is one incoming particle in a volume V the current density is given by

$$j = \frac{v_b}{V} \tag{473}$$

The infinitesimal (differential) cross section

$$d\sigma = \frac{dw_{fi}}{j} = \frac{V|M|^2 d\Phi}{(2E_a V)(2E_b V)} \frac{V}{v_b} = \frac{1}{2m_a 2E_b v_b} |M|^2 d\Phi \tag{474}$$

This formula can be generalized to other frames, too:

$$d\sigma = \frac{dw_{fi}}{j} = \frac{1}{4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2}} |M|^2 d\Phi \tag{475}$$

The full cross section we get by integrating the infinitesimal cross section over the phase space of the final states. It is often useful to partially integrate the cross section, leaving the solid angle of the secondary beam open, e.g.

$$\textit{differential cross section:}\ \frac{d\sigma}{d\cos\theta}\,, \qquad \frac{d\sigma}{d\Omega}(\theta, \phi) \tag{476}$$

$$\textit{full cross section:}\ \sigma = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \frac{d\sigma}{d\Omega}(\theta, \phi) \tag{477}$$

Let's have the following example for a decay process:

$$\mu^-(p) \to e^-(k) + \bar{\nu}_e(q_1) + \nu_\mu(q_2) \tag{478}$$

To the lowest order in perturbation theory a single vertex is responsible for the interaction. Here we use the four fermion interaction model where on the W boson line we simply wrote a $1/M_W^2$:

$$M = \frac{G}{\sqrt{2}} \left[\bar{\nu}_\mu \gamma_\alpha (1-\gamma^5)\mu\right] \left[\bar{\nu}_e \gamma^\alpha (1-\gamma^5) e\right]^* \tag{479}$$

$$M = \frac{G}{\sqrt{2}} \bar{\nu}_\mu \gamma_\alpha (1-\gamma^5)\mu \bar{e}\gamma^\alpha (1-\gamma^5)\nu_e \tag{480}$$

The structure of the Dirac indices can be swapped with the *Fierz* transformation:

$$M = -\frac{G}{\sqrt{2}} \bar{e}\gamma^\alpha (1-\gamma^5)\mu \bar{\nu}_\mu \gamma_\alpha (1-\gamma^5)\nu_e \tag{481}$$

We'll also need the adjoint of the transition amplitude:

$$M^+ = -\frac{G}{\sqrt{2}} \bar{\nu}_e \gamma_\beta (1-\gamma^5)\nu_\mu \bar{\mu}\gamma^\beta (1-\gamma^5) e \tag{482}$$

Then the contribution to the decay rate:

$$|M|^2 = -\frac{G^2}{2} \bar{\nu}_\mu \gamma^\alpha (1-\gamma^5)\mu \bar{\mu}\gamma_\beta (1-\gamma^5) e \bar{e}\gamma_\alpha (1-\gamma^5)\nu_e \bar{\nu}_e \gamma^\beta (1-\gamma^5)\nu_\mu \tag{483}$$

In this form we have ψ and $\bar{\psi}$ types of field appearing next to one an other.

$$|M|^2 = -\frac{G^2}{2}\bar{\nu}_\mu\gamma^\alpha(1-\gamma^5)\mu\bar{\mu}\gamma_\beta(1-\gamma^5)e\bar{e}\gamma_\alpha(1-\gamma^5)\nu_e\bar{\nu}_e\gamma^\beta(1-\gamma^5)\nu_\mu \quad (484)$$

These $\bar{\psi}\psi$ pairs can be regarded as outer products of the solution of the Dirac equation. For the positive energy solution we know:

$$\sum_s u(s,p)\bar{u}(s,p) = \not{p} + m \quad (485)$$

Summing up for the ν_μ spin state means that we are calculating a trace. Let's call the electron's momentum k, the muon's momentum p, and for the electron and muon neutrino: p_1 and p_2, respectively. Neglecting the electron mass and realizing that the muon mass only appears in combinations where the trace of odd powers of gamma matrices and a γ^5 appears, that gives zero. Thus we have

$$\overline{|M|^2} = -\frac{G^2}{2}\mathrm{Tr}\,\not{q}_2\gamma^\alpha(1-\gamma^5)\not{p}\gamma_\beta(1-\gamma_5)\not{k}\gamma_\alpha(1-\gamma^5)\not{p}_1\gamma^\beta(1-\gamma^5) \quad (486)$$

Using gamma matrix identities, such as

$$\gamma^\alpha\not{A}\not{B}\not{C}\gamma_\alpha = -2\not{A}\not{B}\not{C}\,, \qquad \gamma^\alpha\not{A}\not{B}\gamma_\alpha = 4\not{A}\not{B} \quad (487)$$

können wir $|M|^2$ umschreiben:

$$\overline{|M|^2} = 128G^2(pq_1)(kq_2) \quad (488)$$

Now we are evaluating the decay rate formula

$$d\Gamma = \frac{\overline{|M|^2}}{2 \cdot 2m} d\Phi \tag{489}$$

There is an extra factor 2 in the denominator: we ought to have averaged over the incoming spin polarizations (we summed, using the trace).
The electron's energy is E, the muon is at rest. The phase volume:

$$d\Phi = (2\pi)^4 \delta^4(p - k - q_1 - q_2) \frac{d\vec{k}}{2E(2\pi)^3} \frac{d\vec{q}_1}{2\omega_1(2\pi)^3} \frac{d\vec{q}_2}{2\omega_2(2\pi)^3} \tag{490}$$

We first carry out the integral over the neutrio momenta (q_1, q_2):

$$I_{\alpha\beta} = \int q_{1\alpha} q_{2\beta} \frac{d\vec{q}_1}{\omega_1} \frac{d\vec{q}_2}{\omega_2} \delta^4(q_1 + q_2 - q) \tag{491}$$

with $q = p - k$. Clearly, $I_{\alpha\beta}$ is a Lorentz tensor depending on a single external four-momentum q. Thus the index structure may only be proportional to $g_{\alpha\beta}$ or $q_\alpha q_\beta$. In an other orthogonal base:

$$I_{\alpha\beta} = A(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta) + B(q^2 g_{\alpha\beta} - 2q_\alpha q_\beta)\,, \tag{492}$$

Now, A and B must be scalars. Simple calculations yields:

$$A = \pi/6 \qquad B = 0 \tag{493}$$

After itegrating over the neutrino states:

$$d\Gamma = \frac{G^2}{48\pi^4 m}\left[q^2(pk) + 2(qp)(qk)\right]\frac{dk}{E} \tag{494}$$

The electron is ultra-relativistic, we go on with zero electron mass: $k^2 = 0$.

$$qk = (p-k,k) \approx pk = mE\,, \qquad q^2 = (p-k)^2 \approx p^2 - 2pk = m^2 - 2mE \tag{495}$$

First we integrate over the direction of the electron:

$$\begin{aligned} d\Gamma &= \frac{G^2}{12\pi^2 m}(pk)(p^2 - 2pk + 2p^2 - 2pk)EdE \\ &= \frac{G^2}{12\pi^3}(3m^2 - 4mE)E^2 dE \end{aligned} \tag{496}$$

This rate gives the distribution of the electron energies. If we integrate over E we get the full rate, the inverse life time of the muon.

$$\Gamma = \frac{G^2 m^5}{192\pi^3} \tag{497}$$

Mit $m = 0.105\ \mathrm{GeV}$ und $G = 1.17\times 10^{-5}\ \mathrm{GeV}^{-2}$, ist $\Gamma = 0.00069\ \mathrm{GeV} \Rightarrow \Gamma^{-1} = 2.2\times 10^{-6}\ \mathrm{s}$

As a first example we consider a process where an electron is scattered on the electro-magnetic field of an other charged particle, eg. a proton.
The Lagrangian is extended by an extra gauge fixing term, without which we will not be able to solve the Euler Largange equation.

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{\frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu)}_{\mathcal{L}_{\text{gaugefixing}}} - \underbrace{e\bar{\Psi}\gamma^\mu A_\mu \Psi}_{\mathcal{L}_{\text{interaction}}} \qquad (498)$$

The electro-magnetic field obeys its own equation of motion:

$$\partial_\nu \frac{\delta \mathcal{L}}{\delta \partial_\nu A_\mu} = \frac{\delta \mathcal{L}}{\delta A_\mu} \qquad (499)$$

$$\partial_\nu \left[-\partial^\nu A^\mu - \partial^\mu A^\nu) - \frac{1}{\alpha}(\partial_\rho A^\rho) g^{\mu\nu} \right] = -\underbrace{e\bar{\psi}\gamma^\mu \psi}_{J^\mu} \qquad (500)$$

$$\left[-\partial^2 g^{\mu\rho} + \partial^\rho \partial^\mu \left(1 - \frac{1}{\alpha}\right) \right] A_\rho = -J^\mu \qquad (501)$$

We are switching to Fourier space with $\partial_\mu \to -iq_\mu$

$$\underbrace{\left[k^2 g^{\mu\rho} - k^\rho k^\mu \left(1 - \frac{1}{\alpha}\right)\right]}_{M^{\mu\rho}} \tilde{A}_\rho(q) = -\tilde{J}^\mu(q) \tag{502}$$

On the left hand side we have a matrix in μ, ρ, which has to be inverted. It is easy to show, that the inverse D:

$$M^{\mu\rho}(q) D_{\rho\nu} = -g^\mu_\nu \tag{503}$$

$$D_{\rho\nu}(q) = -\frac{g_{\rho\nu} - \frac{q_\rho q_\nu}{q^2}(1-\alpha)}{q^2 + i\epsilon} \tag{504}$$

The Euler-Lagrange equation can then be simply solved:

$$\tilde{A}_\mu(q) = D_{\mu\nu}(q)\tilde{J}^\nu(q) \tag{505}$$

Notice, that in the absence of the α term ($\alpha \to \infty$) the matrix would have had a zero determinant. $\alpha \to 0$ makes a $0/0$ limit in the Lagangian, which forces $\partial_\rho A^\rho \equiv 0$, this is the Lorentz gauge.
Simplest to use: $\alpha = 1$, Feynman gauge

Suppose, a the proton has an incoming p_2 and an outgoing p_4 momentum. Both are particles (not anti-particles). Then we'll use the u solutions with these momenta and insert these into

$$J^\mu(x) = e\bar{\psi}_f(x)\gamma^\mu\psi_i(x) \tag{506}$$

$$J^\mu(x) = e\bar{u}(p_4)\gamma^\mu u(p_2)e^{-i(p_2-p_4)x} \tag{507}$$

Using this as a source to generate the A field, we have in Feynman gauge:

$$A^\mu(x) = e\int d^4q\frac{-g^{\mu\nu}}{q^2+i\epsilon}\delta^4(p_4+q-p_2)e^{-iqx}\bar{u}(p_4)\gamma_\nu u(p_2) \tag{508}$$

In the Euler-Lagrange equation of the electron

$$(i\gamma^\mu\partial_\mu - m)\psi = e\gamma^\mu A_\mu\psi \tag{509}$$

Of course the ψ fields should be the same functions (or operators) on the left and right hand side. However, if we only ask for the leading Taylor expansion in e, we insert the incoming electron the the right hand side, and hope to observe an outgoing particle on the left hand side.

Using the electron propagator S we have

$$\psi_{\rm out}(x) = \int_y S(x-y) e\gamma^\mu A_\mu \psi_{\rm in}(x) \tag{510}$$

This propagator describes the free wave solution from the reaction vertex to the detector. This do not enter into the final probability. The actual transition (S) matrix element is simply

$$S_{fi} = ie \int d^4x' \bar{\phi}_f(x') \gamma^m u A_\mu \phi(x') \tag{511}$$

Here we write for ϕ the u solutions of the Dirac equation, just as we had in the case of the proton. With p_1 incoming and p_3 outgoing electron momentum we have

$$\begin{aligned} S_{fi} &= ie^2 \int \frac{d^4}{(2\pi)^4} \delta^4(p_3 - p_1 - q) \\ & \delta^4(p_4 + q - p_2)(2\pi)^8 \bar{u}(p_3)\gamma_\mu u(p_1) \frac{-g^{\mu\nu}}{q^2 + i\epsilon} \bar{u}(p_{)}\gamma_\nu u(p_2) \end{aligned} \tag{512}$$

An important variant of the same Feynman-Diagram is the $\mu^+\mu^-$ production by electron-positron annihilation. The reaction:

$$\begin{aligned} e^- + e^+ &\rightarrow \mu^- + \mu^+ \\ p_1 + p_2 &= p_3 + p_4 \end{aligned} \tag{513}$$

The momentum of the photon $q = p_1 + p_2$. Now one of the incoming particles is an anti-particle. Thus, when J^μ is constructed we have to use a v solution for the incoming positron. The matrix element:

$$M_{fi} = ie^2 \frac{1}{q^2} \bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(p_3)\gamma^\mu v(p_4) \tag{514}$$

The absolute square will be proportional with the probability of the recation:

$$|M_{fi}|^2 = \left(\frac{e^2}{q^2}\right)^2 L_{\mu\nu} M^{\mu\nu} \tag{515}$$

Now we want to average over the incoming spin and sum over the outgoing spin.

For the electron $L_{\mu\nu}$:

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{2}\sum_{s_1,s_2}\bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(p_1)\gamma_\nu v(p_2) \\ &= \frac{1}{2}\mathrm{Tr}\,\gamma_\mu(\not{p}_1+m)\gamma_\nu(\not{p}_2-m) \end{aligned} \tag{516}$$

For the muon $M_{\mu\nu}$:

$$\begin{aligned} M_{\mu\nu} &= \frac{1}{2}\sum_{s_3,s_4}\bar{u}(p_3)\gamma_\mu v(p_4)\bar{v}(p_4)\gamma_\nu u(p_3) \\ &= \frac{1}{2}\mathrm{Tr}\,\gamma_\mu(\not{p}_4-M)\gamma_\nu(\not{p}_3+M) \end{aligned} \tag{517}$$

At ultrarelativistic energies we can neglect both masses and we have

$$L_{\mu\nu} = 2\left[p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}-(p_1p_2)g_\mu\nu\right] \tag{518}$$

$$M^{\mu\nu} = 2\left[p_3^\mu p_4^\nu+p_3^\nu p_4^\mu-(p_3p_4)g_\mu\nu\right] \tag{519}$$

$$\begin{aligned} L_{\mu\nu}M^{\mu\nu} &= 4\left[2(p_1p_3)(p_2p_4)+2(p_1p_4)(p_2p_4)\right. \\ &\quad \left.-4(p_1p_2)(p_3p_4)+4(p_1p_2)(p_3p_4)\right] \end{aligned} \tag{520}$$

Since $p_1+p_2=p_3+p_4$ the following equalities hold:

$$p_1p_3=p_2p_4 \qquad p_1p_4=p_2p_3 \tag{521}$$

In the CMS frame p_1 and p_2 are pointing to opposite directions, so do p_3 and p_4. If the $\mu^+\mu^-$ pair is flying out under an angle θ relative to the electron-positron beam axis:

$$
\begin{aligned}
p_1 p_3 &= E^2(1-\cos\theta) && (522)\\
p_1 p_4 &= E^2(1+\cos\theta) && (523)\\
q^2 &= (p_1+p_2)^2 = 4E^2 = s && (524)
\end{aligned}
$$

Putting it together:

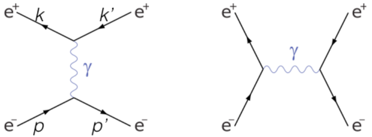
$$
\begin{aligned}
L_{\mu\nu}M^{\mu\nu} &= 8E^4[(1-\cos\theta)^2 + (1-\cos\theta)^2] \\
&= s^2(1+\cos^2\theta), && (525)\\
\overline{|M_{fi}|^2} &= \left(\frac{e^2}{q^2}\right)^2 L_{\mu\nu}M^{\mu\nu} = (4\pi\alpha)^2(1-\cos^2\theta) && (526)
\end{aligned}
$$

The differential and total cross section:

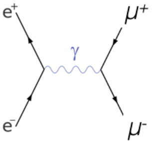
$$
\begin{aligned}
\frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-) &= \frac{\alpha^2}{4s}(1+\cos^2\theta) && (527)\\
\sigma(e^+e^- \to \mu^+\mu^-) &= \frac{4\pi\alpha^2}{3s} && (528)
\end{aligned}
$$

Electron positron experiments:

Elastic (Bhabha) scattering: *(in the massless limit)*



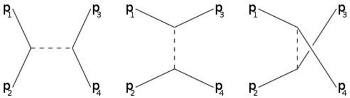
$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)} = \frac{\pi\alpha^2}{s}\left(u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right)$$



$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$$

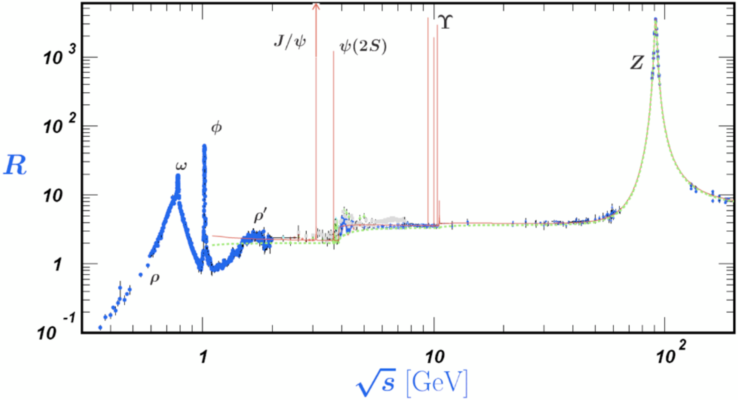
Mandelstam variables:

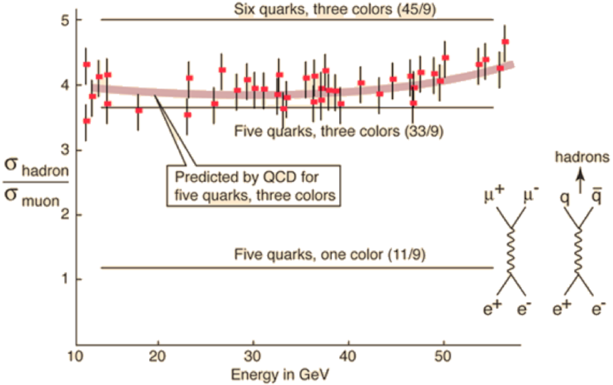
$$s = (p+k)^2, \quad t = (k-k')^2, \quad u = (k-p')^2$$



s-channel t-channel u-channel

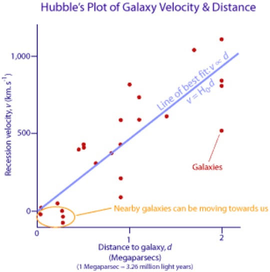
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

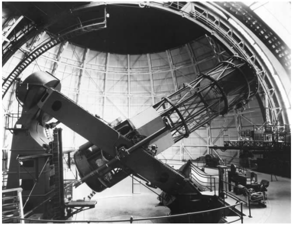




$$R = \frac{\sigma_{hadrons}}{\sigma_{muons}} = \frac{\sum \sigma_{q\bar{q}}}{\sigma_{muons}} = \sum \left(\frac{q_q}{e} \right)^2 \qquad R \approx 3\left[\frac{4}{9} + \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9}\right] = \frac{33}{9}$$

A surprising discovery from 1929





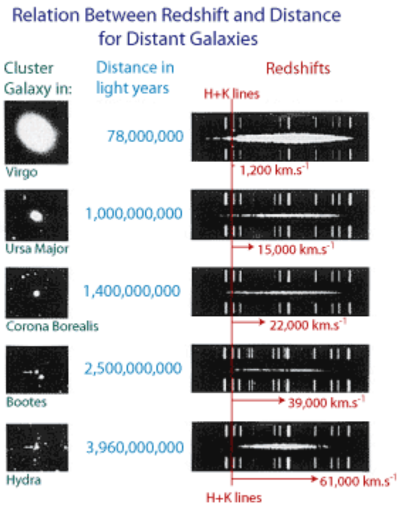
E. Hubble investigated the *redshift* of the light emission from nearby galaxies as a function of distance.
The distance he got from luminosities of Cepheid variable stars. The brightness of these were already known to be related to their periodicity.

The *redshift* is understood from the relativistic Doppler effect:

$$\lambda' = \lambda\sqrt{\frac{1+v/c}{1-v/c}} = \lambda(1+z) \tag{529}$$

The redshift parameter $z = \Delta\lambda/\lambda$ is zero if the object is at rest (relative to us), positive if the object is moving away.
The redshift is measured by taking the spectrum of the galaxy. This normally consists of several emission lines, where each line is shifted consistently with a factor z from the expected spectrum.

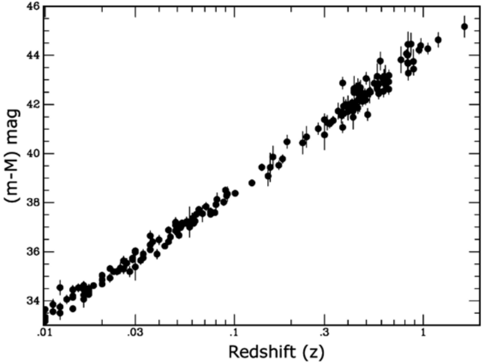
For nearby galaxies $z \lesssim 1$, but various objects have been found between $z = 6 \ldots 11$.



Hubble used Vesto Slipher's velocity measurements.

A more recent plot: Hubble diagram for type Ia supernovae to $z \approx 1$.

Type Ia supernovae are standard candles: well defined brightness



(by R.P. Kirschner PNAS 2004;101:8-13)

What we have learned can be summarized as

$$v = Hr \tag{530}$$

Or, in other words, everything is moving away, with a velocity proportional to the distance. Thus one can introduce a scale factor $R(t)$

$$H = \frac{\dot{R}}{R} \tag{531}$$

Today, we know that H was not constant in the earlier Universe, and today

$$H_0 = 100 h_0 kms^{-1} Mpc^{-1} \tag{532}$$

$$h_0 = 0.678(9) \qquad \text{Plack 2015} \tag{533}$$

with $1Mpc = 3.09 \cdot 10^{19} km$ (*megaparsec*).
For a long time h_0 was not known so precisely. But even today, it is most convenient to express distance in z.
From distant objects light takes a long time to arrive to us. Thus z is also an inticator of the age of the seen object. At large redshift:

$$t(z) \approx \frac{2}{3H_0(1+z)^{3/2}} \tag{534}$$

A more precise formula also gives the age of the Universe: ≈ 13.8 Gyr.

This universal expansion is understood in the framework of general relativity. The *Einstein* equation relates the

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{535}$$

where
$T_{\mu\nu}$ is the energy momentum tensor of matter
$R = g^{\alpha\beta} R_{\alpha\beta}$ and
$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ is the Ricci tensor
and the Riemann-tensor is defined as
$R^{\mu}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho}$
$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$ is the Christoffel symbol.
By Λ one may introduce a cosmological constant, which plays a role in Today's Universe, it was negligible in the early Universe relative to the then high energy densitiy.
Thus, we have a second order equation for the $g_{\mu\nu}$ metric tensor, which is now space and time dependent.

There are several approaches how to look for specific solutions to Einstein's equation:
– solve around a singularity (e.g. Schwarzschild solution)
– linearize the equation round the $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$ Minkowski-space metric tensor:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \tag{536}$$

this will lead to

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} + \partial_\nu\partial_\sigma h_{\mu\rho}\right) \tag{537}$$

and with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}h_{\sigma\rho}$

$$\partial^2\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{538}$$

There is a "gauge symmetry" in the Eistein equation, too. By selecting the analog of a Lorentz gauge one finds

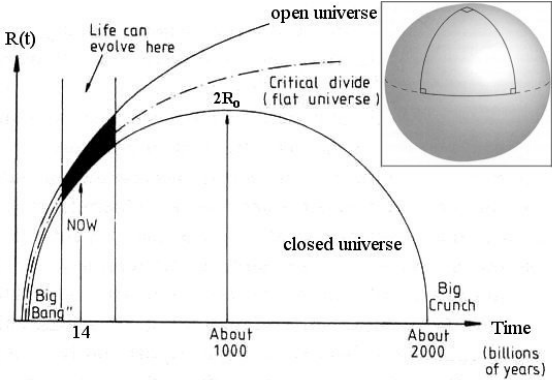
$$\partial^2\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{539}$$

This equation describes gravitational waves.

Now, we are seeking a very different kind of solution: one with spatial homogeneity and isotropy.
The Friedmann-Robertson-Walker metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{540}$$

This is a set of three Ansätze, with $k = -1, 0, +1$ for a closed, flat, or open Universe, respectively.



Let's assume that the Universe is filled with matter, that has a homogeneous pressure p and energy density ρ.

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix} \tag{541}$$

The trace yields:

$$T^{\mu}_{\nu} = -\rho + 3p \tag{542}$$

By calculating the four divergence of the zeroth column we can arrive at

$$\begin{aligned} 0 &= \nabla_\mu T^\mu_0 && (543) \\ &= \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu 0} T^0_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda && (544) \\ &= -\partial_0 \rho - 3\frac{\dot{a}}{a}(p+\rho) && (545) \end{aligned}$$

The relation of p and ρ is shorly referred to, as the equation of state of the matter (or the whole Universe). E.g.

$$p = w\rho \tag{546}$$

For the specific Ansatz the Einstein equation gives:

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) \qquad \text{component}\mu,\nu = 0,0 \quad (547)$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) \qquad component\,\mu,\nu = i,j \quad (548)$$

Where we used the explicit form of the energy momentum tensor in terms of p the pressure and ρ the energy density.
From a combination of the two components of the Einstein equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (549)$$

Hubble's parameter naturally appar in the right hand side as

$$H = \frac{\dot{a}}{a} \quad (550)$$

Thus, if the density of the Universe happens to be smaller (larger) than the critical density ($\Omega_{\text{crit}} = 1$), the open (closed) solution will realize.
The Friedmann equation has an obvious initial singularity $a = 0$ at $t = 0$. This singularity is the *Big Bang*.

There is a critical energy density:

$$\rho_{\rm crit} = \frac{3H^2}{8\pi G}\,, \tag{551}$$

the actual density is often expressed in units of $\rho_{\rm crit}$:

$$\Omega = \rho/\rho_{\rm crit}\,. \tag{552}$$

The Friedmann equation in terms of Ω:

$$\Omega - 1 = \frac{k}{H^2 a^2} \tag{553}$$

The density is, of course, not constant. From the energy conservation equation (545) we have

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}\,, \tag{554}$$

thus

$$\rho(t) \sim [a(t)]^{-3(1+w)} \qquad \text{where} \quad w = p/\rho \tag{555}$$

By direct use of the Friedmann equation

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} = -3(1+w)\sqrt{\frac{8\pi G\rho}{3}} \tag{556}$$

Upon integration we get for the energy density

$$\rho(t) = \frac{3c^2}{8(1+w)\pi G}\frac{1}{t^2} \tag{557}$$

For a relativistic gas (e.g. photon gas), $w = 1/3$, or

$$\rho = \frac{\pi^2 (kT)^4}{15\pi\hbar^3 c^3} \tag{558}$$

This results in

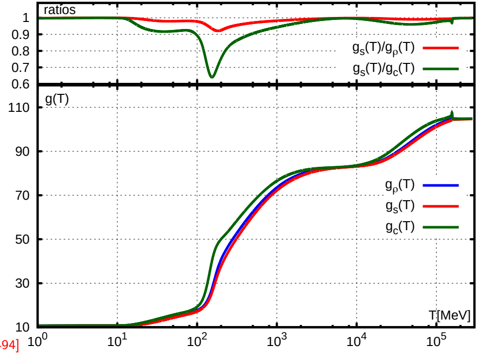
$$T \approx \frac{10^{10} K}{t^{1/2}_{\mathrm{in\ seconds}}} \tag{559}$$

In 1965 Penzias and Wilson hase observed a black body radiation coming from the depths of space, the *cosmic microwave background* radiation. Its spectrum corresponds to the Plack black body radiation, thus a temperature can be associated. It is: $T_{CMB} = 2.73 \pm 0.01K$

If we have $w = 1/3$ for the Universe, it is said to be *radiation dominated*. If the Universe had always been radiation dominated, this would give ~ 10 Gyr for the age of the Universe.

Adding up QCD, the free light particles and the electroweak theory:

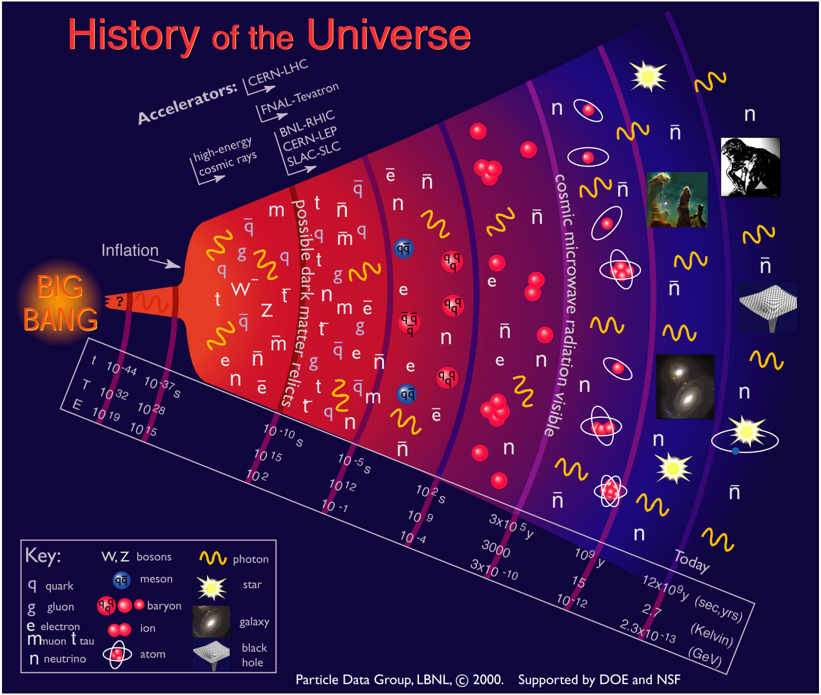
number of effective degrees of freedom:



[Wuppertal-Budapest 1606.07494]

energy dens. $\rho = g_\rho \dfrac{\pi^2}{30} T^4$ entropy dens. $s = g_s \frac{2\pi^2}{45} T^3$ heat cap. $c = g_c \dfrac{2\pi^2}{15} T^3$

cooling rate in early universe $\dfrac{dT}{dt} = -\dfrac{T^3}{M_{pl}} \dfrac{2\pi^{3/2}}{3\sqrt{5}} \dfrac{\sqrt{g_\rho} g_s}{g_c}$


History of the Universe
Accelerators:
CERN-LHC
FNAL-Tevatron
BNL-RHIC
CERN-LEP
SLAC-SLC
high-energy cosmic rays
Inflation
BIG BANG
possible dark matter relicts
cosmic microwave radiation visible
t 10^{-44} 10^{-37} s 10^{-10} s 10^{-5} s 10^{2} s 3×10^{5} y 10^{9} y 12×10^{9} y (sec, yrs)
T 10^{32} 10^{28} 10^{15} 10^{12} 10^{9} 3000 15 2.7 (Kelvin)
E 10^{19} 10^{15} 10^{2} 10^{-1} 10^{-4} 3×10^{-10} 10^{-12} 2.3×10^{-13} (GeV)
Today
Key:
q quark
g gluon
e electron
m muon t tau
n neutrino
W, Z bosons
meson
baryon
ion
atom
photon
star
galaxy
black hole
Particle Data Group, LBNL, © 2000. Supported by DOE and NSF

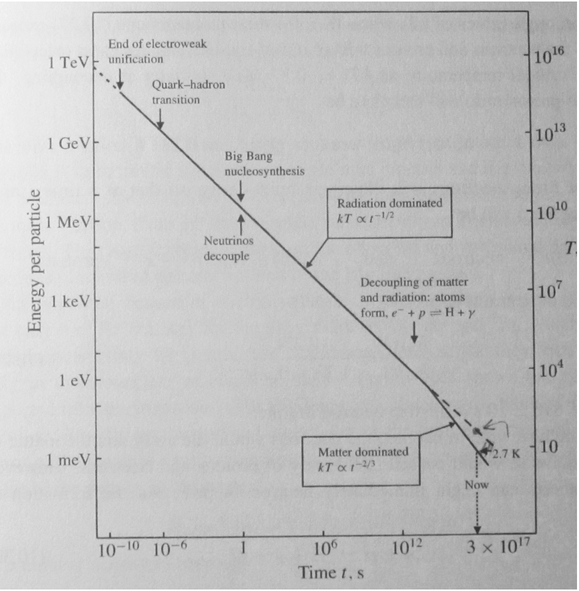
Mostly, the expansion of the Universe is slow enough for the cooling to take place simultaneously for all degrees of freedom.
However, there are several exceptions. E.g. the rate of weak interactions starts to compete with the expansion as the temperature drops to 3 MeV.
At this instant (cca. 0.1-1 s after Big Bang) the equilibrizing reaction

$$e^{+} + e^{-} \leftrightarrow \nu + \bar{\nu} \tag{560}$$

stops. Neutrinos fly off with large momentum without ever having the chance to thermalize again with the cooling matter.
As the Universe's temperature dropped even further (0.3 eV), the photons, too, fell out of equilibrium. For the reaction

$$e^{-} + p \leftrightarrow \gamma + H + Q \tag{561}$$

(with $Q = 13.6$ eV being the binding energy in a Hydrogen atom) stops, because the energetic photons that would deliver the energy Q are missing. This means that all electrons pair with a protons and vice versa, and the Universe is basically made up of neutral particles.
This moment is the decoupling of radiation (at about 3×10^{6} years).
At about the same time the energy density of the Universe was dominated by non-relativistic degrees of freedom, making $w \approx 0$, and that has changed the course of expansion to $T \sim t^{-2/3}$.



For later times the equation of state changed to $w = 0$ *matter dominated*.
Then the temperature followed $T \sim t^{-2/3}$.

At about ~ 1 s after Big Bang, the Universe was dominated by leptons, photons, neutrons and protons. Their relative number was controlled by the reactions

$$\begin{aligned} \nu_e + n &\leftrightarrow e^- + p \\ \bar{\nu}_e + p &\leftrightarrow e^+ + n \\ n &\rightarrow p + e^- + \bar{\nu}_e \end{aligned} \tag{562}$$

The temperature (and also the typical momenta) are much lower than the proton's and neutron's rest mass, these are, thus, non-relativistic. Their relative weight is determined by the Boltzmann factor:

$$\frac{N_n}{N_p} = \exp\left(\frac{-Q}{kt}\right) \qquad Q = (M_n - M_p)c^2 = 1.293\ \mathrm{MeV} \tag{563}$$

At about $kT = 0.87$ MeV the rate for the weak interactions drops (the decay times will exceed the age of the Universe), the protons and neutrons freeze out. with the constant rate of

$$N_n/N_p = \exp(-Q/kT) = 0.23 \tag{564}$$

Now, the neturons can still decay into protons at a rate of $1/\tau \approx 1/896s^{-1}$:

$$\frac{N_n(t)}{N_p(t)} = \frac{0.23e^{-t/\tau}}{1.23 - 0.23e^{-t/\tau}} \tag{565}$$

and the neutrons would die away if nothing else would happen. However, as soon as neutrons appear, the nucleosynthesis can appear

$$n + p \leftrightarrow {}^2H + \gamma + Q \tag{566}$$

with $Q = 2.22$ MeV. This is a mainly electromagnetic process, and will keep up thermal equilibrium, unlike the weak processes. By $kT = 0.05$ MeV there will have been no energetic photons left to maintain the reactions, and the photodisintegration of the deuterium stops. Then competing reactions leading to helium production take over:

$$\begin{aligned} {}^2\mathrm{H} + n &\rightarrow {}^3\mathrm{H} + \gamma \\ {}^3\mathrm{H} + p &\rightarrow {}^4\mathrm{He} + \gamma \\ {}^2\mathrm{H} + p &\rightarrow {}^3\mathrm{He} + \gamma \\ {}^3\mathrm{He} + n &\rightarrow {}^4\mathrm{He} + \gamma \end{aligned} \tag{567}$$

For $kT = 0.05$ MeV, corresponding to an expansion time of $t \approx 400$ s, the neturon-to-proton ratio

$$r = \frac{N_n}{N_p} = 0.14 \tag{568}$$

Neutrons inside of nucleons will not decay, thus the ratio is fixed.

Can r of today be observed?

The ratio follows:

$$Y = \frac{4N_{\mathrm{He}}}{4_{\mathrm{He}} + N_{\mathrm{H}}} = \frac{2r}{1+r} = 0.25 \tag{569}$$

which is a mass fraction of the Helium and Hydrogen. This could be measured at many different celestial sites (including our solar system), and found to be

$$r_{\mathrm{exp}} = 0.24 \pm 0.01 \tag{570}$$

This agreement boosted the confidence in the Big Bang theory.

- The matter density can be estimated e.g. by summing up the mass of stars, gas, dust etc. One finds

$$\rho_{\mathrm{lum}} \approx 2 \cdot 10^{-29}\ \mathrm{kg\ m}^{-3} \tag{571}$$

or, equivalently

$$\Omega_{\mathrm{lum}} \approx 0.0063 \tag{572}$$

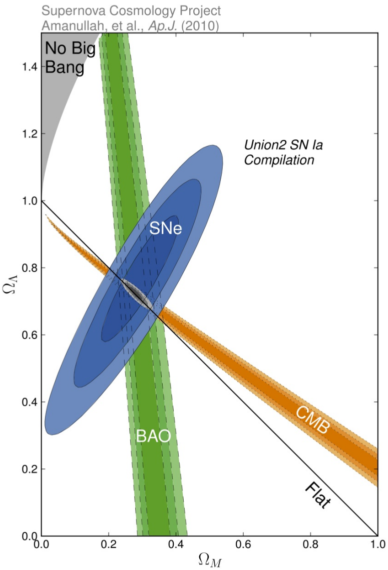
- If we count all baryons that must have been created by the model of baryogenesis in the early Universe, this number increases to

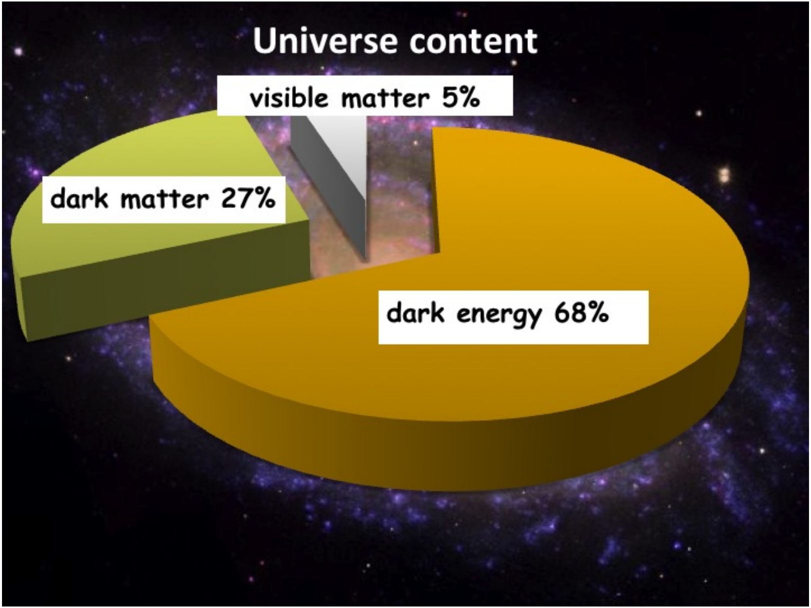
$$\Omega_{\mathrm{baryon}} \approx 0.02 \ldots 0.06 \tag{573}$$

- If, however, we total mass is inferred from the observed galactic rotation curves, a much higher value emerges:

$$\Omega_{\mathrm{matter}} \gtrsim 0.3 \tag{574}$$

1.) Most baryonic matter is non-luminous
2.) Most matter is non-baryonic


Supernova Cosmology Project
Amanullah, et al., Ap.J. (2010)
No Big Bang
Union2 SN Ia Compilation
SNe
BAO
CMB
Flat
Ω_Λ
Ω_M
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4


Universe content
visible matter 5%
dark matter 27%
dark energy 68%

What could make up the large non-baryonic matter content of the universe?

- Obvious canditates could be the neutrinos. These are known to have decoupled and stay on.
  If neutrinos decoupled at $kT \approx 3$ MeV, they have cooled down by now to 1.9 K. (This is a $(4/11.)^{1/3}$ factor lower than the known 2.7K temperature of the photonic background, the latter was boosted by the $e^+, e^- \to \gamma$ annihilation process.)
  From the temperature sizeable contribution ($\Omega \sim 1$) could arise, if a small mass is assumed ($\sim 10^{-8} M_p \sim \mathcal{O}(10)$ eV).
  This type of dark matter is **hot dark matter**. It would stream rapidly under gravity and tend to iron out any primordial density fluctuations. Thus, it cannot seed the forming of the observed structures in the Universe (e.g. galaxies). For this reason *Cold Dark Matter* hypoteses are favored.

- Other candidates are WIMPS
  Weakly interacting massive particles.
  – These are decoupled, massive, good dark matter candidates by construction.
  – Since all particles were created equally in the Big Bang, their abundance today is determined by the annihilation cross section.
  – Colud be light (or not so light) superpartners (neutralino).
  – Should be fould in experimens, if they exist.
  – Cosmic WIMPS at the GeV scale (assuming a typical $v/c = 0.001$), could be detected via elastic scattering from nuclei, where the low energy nuclear recoil is detected.

The Standard Model breaks the CP symmetry.

QCD could break this symmetry with a CP-odd term: $q(x) = \frac{g^2}{64\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu}$

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + i\bar{\psi}(D_\mu \gamma^\mu - m)\psi + \theta \frac{g^2}{64\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu}$$

$$Z = \int \mathcal{D}\mathcal{A}\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S^E_{QCD}} e^{i\theta Q} \sim \sum_Q P(Q) e^{i\theta Q}$$

Q is the winding number of gauge the configuration.

Q is integer and *topological*

$$Q = \int d^4 q(x)$$

Constraint form the neutron electric dipole moment: $|\theta| < 10^{-10}$

Fine tuning?

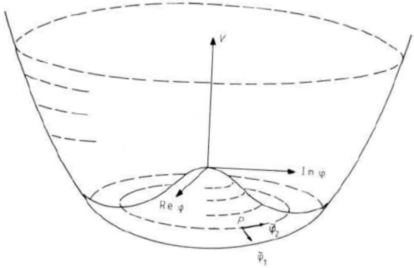
If at least one quark mass is zero a $U(1)_A$ rotation can erase θ.

[Peccei-Quinn 1977]: new particle field

A variant: the KSVZ (Kim-Shifman-Vainstein-Zakharov) axion model

$$\mathcal{L}(\Phi,\Psi) = \partial_\mu\Phi^*\partial_\mu\Phi + V(\Phi^*\Phi) + \Phi\bar{\Psi}_L\Psi_R + \Phi^*\bar{\Psi}_R\Psi_L + \bar{\Psi}D(A)\Psi + \mathcal{L}_{QCD}(A)$$

where ψ is a heavy ($m \sim f_A$) fermion with color charge.
The potential V is such that a spontaneous breaking of the $U(1)$ symmetry occurs at $T \sim f_A$.



$$\Phi = (f_A + r)e^{i\theta}, \qquad \phi/f_a = \theta$$

The emerging pseudo-Goldstone boson (ϕ) is called **axion**.
The potential is tilted according to θ, the axion will roll to the minimum.

Effective Lagrangian:

$$\mathcal{L} = \partial_\mu \phi \partial_\mu \phi + i \frac{\phi}{f_A} q + \partial_\mu \theta \cdot (\ldots) + \mathcal{L}_{QCD}$$

$$\frac{\partial V_{eff}}{\partial \phi} = -\frac{1}{f_A} \frac{g^2}{64\pi^2} \langle F^a_{\mu\nu} \tilde{F}^{a,\mu\nu} \rangle$$

$$m_\phi^2 = \frac{\partial^2 V_{eff}}{\partial \phi^2} = \frac{1}{f_A^2} \int d^4x \langle q(x) q(0) \rangle = \frac{1}{f_A^2} \chi_t$$

Axion mass: $f_A^2 m_A^2(T) = \chi(T)$, purely QCD quantity.
$\chi(T = 0)$ is known from chiral perturbation theory:

$$\chi(T = 0) = [75.6 * (1.8)(0.9) MeV]^4$$

Thus $m_A|_{\mathrm{today}} f_A \approx 0.0057 GeV^2$.

Once $\chi(T) = f_A^2 m_A^2(T)$, $\varepsilon(T)$, $s(T)$ available:

Axion equations of motion + Einstein equations

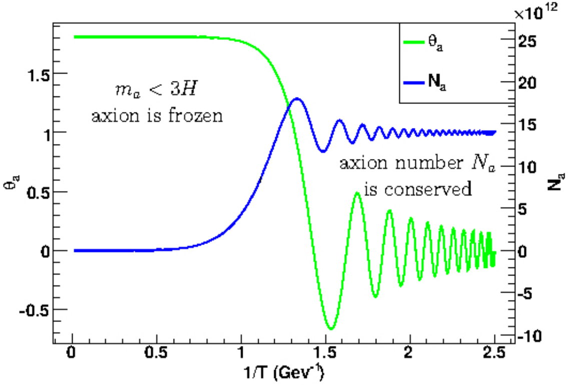
$$\frac{d^2\theta}{dt^2} + 3H(T)\frac{d\theta}{dt} + m_A^2(T)\frac{d}{d\theta}(1 - \cos\theta) = 0$$

$$H^2(T) = \frac{8\pi}{3M_{pl}^2}\varepsilon(T)$$

$$\frac{d\varepsilon}{dt} = -3H(T)s(T)T$$

The θ field cannot roll into the pontential minimum, because of the high "friction" H.
Key assumption: $\theta(x) = \theta(t)$ spatially constant $\rightarrow$
no strings, domain walls, etc

The PQ symmetry breaks spontaneously: $\theta = \theta_0$.
Initially the axion is massless ($T \gg T_{QCD}$, $m_A^2 \sim \chi_t$).
The axion starts rolling when $3H(T) \leq m_A(T)$, this T is called T_{osc}.



The evolution is adiabatic: $\mathrm{entropy}/n_{\mathrm{axion}}$ is constant:

$$n_{A,\mathrm{today}} = \frac{n_A(T_{osc})}{s(T_{osc})} s_{\mathrm{today}}$$

What is the initial angle θ_0? It depends on these scenarios:
Pre-inflation scenario: $f_A \sim T_{PQ}$ is above the reheat temperature, the angle can by anything...
Post-inflation scenario: all angles are present, one has to average. This gives effectively $\theta_0 = 2.155$.

If this mechanism is responsible for all dark matter: $m_A = 28(2)\mu eV$
If this mechanism is responsible for 50% dark matter: $m_A = 50(4)\mu eV$
If this mechanism is responsible for 1% dark matter: $m_A \sim 1500\mu eV$
Rest from defects (strings, domain walls, etc) within axion picture, potentially more types of DM also